INDAM MEETING:

HYPERBOLIC DYNAMICAL SYSTEMS IN THE SCIENCES

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Infinitely many cascades must exist as chaos arises in $\mathbb{R}^n$

Evelyn Sander and I have established a general theory of why period-doubling cascades exist (in n dimensions), including why smooth systems have infinitely many cascades. Feigenbaum's results describe how a cascade's bifurcations scale—if the cascade exists. Collet, Eckmann, and Lanford extended these results by 1980 to argue that almost every cascade of sufficiently smooth maps have Feigenbaum-like scaling. They did not—except in special cases—show that cascades exist. We fill that gap using topological arguments.

We show that infinitely many cascades must exist as a system goes from having only finitely many periodic orbits to having chaotic dynamics. Our theory is for generic smooth one-parameter maps $F(\mu, x)$ where x is n-dimensional. Here is one corollary for maps with horseshoes in dimension 2 such as the time-1 map of the forced damped pendulum or double-well Duffing equation.

The Route to Chaos Theorem Assume $F(\mu, x)$ is smooth and x is two-dimensional. Under additional mild restrictions, if there are parameter values μ_0 and μ_1 for which $F(\mu_0, \cdot)$ has at most a finite number of periodic orbits, and $F(\mu_1, \cdot)$ has exponential growth of number of periodic orbits as a function of the period. Then there are infinitely many period-doubling cascades between μ_0 and μ_1.

The above result also holds when x is one dimensional with minor wording changes. In addition we have discovered a new phenomenon in which there are paired cascades, that is, two cascades that are connected by a path of periodic orbits. The quadratic map $\mu - x^2$ has no paired cascades but almost all cascades are paired for the forced damped pendulum and for the forced single and double-well Duffing equations.