“Anisotropic Sobolev spaces adapted to piecewise hyperbolic dynamics

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(Results announced in Beijing, August 2009)

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(Mittag-Leffler, March 30, 2010)

Science:

The papers:

Good Banach spaces for piecewise hyperbolic maps via interpolation (with S. Gouëzel)
Annales de l'Institut Henri Poincaré / Analyse non linéaire, 26 (2009) 1453-1481 (BG1)

Banach spaces for piecewise cone hyperbolic maps (with S. Gouëzel)
J. Modern Dynamics, 4 (2010) 91-137 (BG2)

The setting:

Discrete-time piecewise hyperbolic dynamics

Science fiction:

The medium-term goal:

Proving exponential decay of correlations for the continuous-time Sinai billiard (smooth dispersive obstacles, finite horizon, dimension two) and Hölder observables

Science fiction:

The long-term goal:

Proving exponential decay of correlations for the continuous-time Sinai billiard (smooth dispersive obstacles, finite horizon, dimension two) and Hölder observables

NB: stretched exponential upper bounds are known (Chernov, 2007)

The short/medium-term goals:

1) Exponential decay of correlations for piecewise hyperbolic contact flows
(WIP with Carlangelo Liverani)

2) A new proof of exponential decay of correlations for discrete-time 2D Sinai billiards
(WIP with Péter Bálint and Sébastien Gouëzel)

Functional approach:

(No Markov partition, no symbolic dynamics, no reduction to expanding setting)

Why?

-For discrete-time systems, finite or countable Markov partitions give exponential decay of correlations in many hyperbolic and nonuniformly hyperbolic situations, also with singularities (exponential decay for 2-D Sinai billiard, L.S.Young, Ann. of Math, 1998)

-For continuous-time, Dolgopyat's argument gives exponential decay for certain smooth hyperbolic systems (Ann of Math, 1998) with finite Markov partitions.

-It is possible to extend Dolgopyat's argument to simple suspensions of systems with countable Markov partitions (Baladi-Vallée, Proc AMS, 2005)

-This extension was useful to prove exponential decay of correlations for Teichmuller flows (Avila-Gouëzel-Yoccoz, Publ Math IHES, 2006)

-However, it cannot be combined with Young's construction to get exponential decay of correlations for continuous-time Sinai billiards (the metric in the Young tower is dynamical, the connection with euclidean metric is lost, and therefore the lower bounds coming from the contact property in the original space cannot be exploited)

Why a functional approach?

-For continuous-time, Dolgopyat's argument gives exponential decay for certain smooth hyperbolic systems (Ann of Math, 1998) with finite Markov partitions.

-It is possible to extend Dolgopyat's argument to simple suspensions of systems with countable Markov partitions (Baladi-Vallée, Proc AMS, 2005)

-However, this extension cannot be combined with Young's construction to analyse correlations of continuous-time Sinai billiards

-For continuous-time systems, Liverani modified Dolgopyat's argument to give exponential decay in the contact smooth hyperbolic case (Ann of Math, 2004). His argument does not use symbolic dynamics, but it requires a good Banach space for the hyperbolic transfer operator, i.e. a functional approach. Liverani introduces some averaging operators which must behave well with respect to the norm.

NB: Tsujii has yet another argument, which also requires a good Banach (in fact, Hilbert!) space for the hyperbolic transfer operator

Science fiction:

A four-step plan to reach the final goal:
(exponential decay of correlations for continuous time Sinai billiards)

Strategy:

Step 1: find a good Banach space for piecewise hyperbolic systems
(Baladi-Gouëzel 2009, Baladi-Gouëzel 2010)
challenge: discontinuities ✔

NB: Demers-Liverani 2008 were the first to construct such a space

Step 2: upgrade the norm to exploit Liverani's version of Dolgopyat's computation to continuous-time (contact) piecewise hyperbolic systems
(WIP with Liverani)
challenge: the cancellation argument

Step 3: find a good Banach space for piecewise hyperbolic systems with blowup of derivatives along some boundaries, to include discrete-time billiards, getting a new proof of Young's result
(WIP with Balint and Gouëzel)
challenge: need to work with homogeneity layers ?

Step 4: combine the ingredients of Steps 2 and 3 to reach the final goal (exponential decay of correlations for continuous-time 2D Sinai billiards with finite horizon)

Back to Science (i.e., theorems):

A) The functional aproach in the discrete-time smooth hyperbolic case
(a brief reminder - 2002-2008)

B) The functional aproach in the discrete-time piecewise smooth hyperbolic case

«Step 1: find a good Banach space for piecewise hyperbolic systems
(Baladi-Gouëzel 2009, Baladi-Gouëzel 2010)
challenge: discontinuities»

A) The functional aproach in the discrete-time smooth hyperbolic case (a brief reminder - 2002-2008)

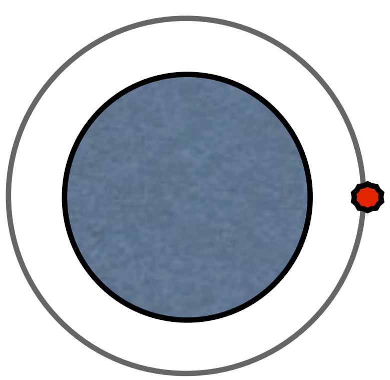
Theorem: Let M be a compact Riemann manifold, let T be a $C^{1+\alpha}$ Anosov diffeomorphism with a dense orbit. Put

$$\mathcal{L}\phi = \frac{\phi \circ T^{-1}}{|\det T \circ T^{-1}|}, \quad \phi \in C^\infty .$$

Then there exists a Banach space $\mathcal{B}$ of distributions on M so that

$$C^\infty(M) \subset \mathcal{B}, \quad L^\infty \cap B \text{ is dense in } \mathcal{B},$$

and $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ has "Perron-Frobenius spectrum," that is, a spectral gap:



The essential spectral radius of $\mathcal{L}$ is < 1,
the spectral radius is $= 1$ and 1 is a simple eigenvalue and the only eigenvalue of modulus 1.

References:

Blank-Keller-Liverani, Nonlinearity, 2002 (suboptimal bound on essential spectral radius)

Baladi, 2005 (Sobolev approach, assuming smooth foliations) (B2005)

Gouëzel-Liverani, ETDS 2006, J Diff Geom 2008 (GL1-2)

Baladi-Tsujii, Ann Inst Fourier 2007, Brin proceedings 2008 (Sobolev approach, BT1-2)

Consequences:

Note that $\mathcal{L}^*(dx) = dx$. It follows (by standard but lengthy arguments) that the fixed point $\phi_0 = \mathcal{L}(\phi_0)$ of $\mathcal{L}$, normalised by $\phi_0(1) = 1$ is in fact a T-invariant probability measure, noted μ, which is the unique SRB (physical) measure of T, that is, for Lebesgue almost every $x \in M$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k(x)} = \mu \quad \text{(weak $*$-convergence)}.$$

The spectral gap property implies in particular exponential decay of correlations (for Holder observables) of the SRB measure.

B) The functional aproach in the discrete-time piecewise smooth hyperbolic case

«Step 1: find a good Banach space for piecewise hyperbolic systems
(Baladi-Gouëzel 2009, Baladi-Gouëzel 2010)
challenge: discontinuities»

The new problem: What if the dynamical system T is only *piecewise* smooth and hyperbolic?

*The Banach spaces of GL1, GL2, BT1, BT2 are not invariant under the action of the transfer operator!

*Demers-Liverani (TAMS 2008) introduced a Banach space on which the transfer operator has a spectral gap for surface piecewise smooth and hyperbolic maps

(Unfortunately, adapting this Banach space to discrete-time billiards seems to be very labor intensive task.)

The new problem: What if the dynamical system T is only *piecewise* smooth and hyperbolic?

«Characteristic functions need to be bounded multipliers»

In the smooth hyperbolic case

$$\mathcal{L}\phi = \frac{\phi \circ T}{|\det DT \circ T^{-1}|}\,.$$

In the piecewise smooth hyperbolic case ($M = \cup_i \Omega_i$ with $T_i = T|_{\Omega_i}$ smooth)

$$\mathcal{L}\phi = \sum_i \chi_{\Omega_i} \frac{\phi \circ T_i}{|\det DT_i \circ T_i^{-1}|}\,.$$

Therefore:

Additional requirement for the Banach space $\mathcal{B}$: for any nice enough domain Ω, there should exist C_Ω so that

$$\|\chi_\Omega \phi\|_{\mathcal{B}} \le C_\Omega \|\phi\|_{\mathcal{B}} \quad \forall \phi \in \mathcal{B}\,.$$

*This «bounded multiplier» property is *violated* by the norms in the articles GL1-GL2, BT1-BT2.

*But: we were able to prove in BG1 that this property is *satisfied* by the norms in B2005 (smooth foliations), that we shall recall next:

Isotropic and anisotropic Sobolev spaces:

Isotropic (ordinary) Sobolev norm in $\mathbb{R}^d$ $(d \geq 1)$: fix $1 < p < \infty$, $t \in \mathbb{R}$,

$$\|\phi\|_{H^t_p(\mathbb{R}^d} = \|\mathbb{F}^{-1}\big((1+|\xi|^2)^{t/2}\mathbb{F}\phi\big)\|_{L^p(\mathbb{R}^d)} ,$$

where $(\mathbb{F}\phi)(\xi)$ is the (continuous) Fourier transform of $\phi(x)$.

Anisotropic Sobolev norm in $\mathbb{R}^d$ $(d \geq 2,\ d = d_s + d_u)$: fix $1 < p < \infty$, $s, u \in \mathbb{R}$ (say $s < 0$, $s + u > 0$)

$$\|\phi\|_{H^{u,s}_p(\mathbb{R}^d)} = \|\mathbb{F}^{-1}\big((1+|\xi|^2)^{s/2}(1+|\xi|^2+|\eta|^2)^{u/2}\mathbb{F}\phi\big)\|_{L^p(\mathbb{R}^d)} ,$$

where $(\mathbb{F}\phi)(\xi, \eta)$ is the (continuous) Fourier transform of $\phi(x, y)$, with $\eta, y \in \mathbb{R}^{d_u}$ and $\xi, x \in \mathbb{R}^{d_s}$.

Lemma (BG1): For any $d \geq 2$, for any $1 < p < \infty$, for any $u, s \in \mathbb{R}$ so that

$$\frac{1}{p} - 1 < s \leq 0 \leq u < 1/p\,,$$

for any $L \geq 1$, and for each $\Omega \subset \mathbb{R}^d$ for which there exists a system of coordinates so that almost each line parallel to the axes intersects Ω into at most L connected components, the function χ_Ω is a bounded multiplier on $H_p^{u,s}(\mathbb{R}^d)$.

NB: the condition on the domain will give rise to transversality conditions later on

Proof:

Case 1: $s = 0$ and $u \geq 0$ (isotropic Sobolev space $t = u$). Strichartz (1967) found the necessary and sufficient condition

$$0 \leq t < \frac{1}{p}\,.$$

(Heuristic argument: $\mathbb{F}(sgn) \sim \xi^{-1}$, $\mathbb{F}^{-1}(|\xi|^{t-1}) \sim |x|^{-t}$, $\int_{-a}^{a} |x|^{-tp}\,dx < \infty$ if and only if $tp < 1$.)

Case 2: $s = 0$ and general u. Combining Strichartz and duality gives

$$-1 + \frac{1}{p} < u < \frac{1}{p}\,.$$

Case 3: general case. This follows from Case 2, Fubini, and an old complex interpolation result of Triebel:

$$H_p^{u,s} = [H_{p_1}^{u_1,s_1}, H_{p_2}^{u_2,s_2}]_\theta\,, \quad \forall 0 \leq \theta \leq 1$$

where $u = u_1^\theta \cdot u_2^{1-\theta}$, $s = s_1^\theta \cdot s_2^{1-\theta}$, and $1/p = \theta/p_1 + (1-\theta)/p_2$.

QED.

*The previous lemma is the key to BG1 («Good Banach spaces for piecewise hyperbolic maps via interpolation»), the main result of which is a bound on the essential spectral radius of the transfer operator acting on the Banach space obtained by viewing anisotropic Sobolev spaces in charts (using a partition of unity).

☞ Beware! The upper bound involves «complexity» numbers, which quantify how fast the dynamics cuts up ✄ the space under iteration. The bound is <1 only if the hyperbolicity dominates the complexity, and the restrictions on parameters $s<0<u$ are a limiting factor here (this is a «fact of life», however).

☹ In order to trivialise the stable manifolds via continuously differentiable charts, we needed to assume in BG1 that the system admitted a (piecewise) differentiable invariant stable foliation. As the invariant stable foliation is in general only measurable, this is a rather restrictive assumption!

☺ The other assumptions of BG1 are mild: finitely many pieces, a Hölder derivative on each piece, a weak transversality assumption (between the boundaries of the pieces and the invariant stable foliation), unstable cones.

BG2 replaces the (very strong) assumption of the existence of a differentiable stable foliation in BG1 by:

*stable cones;

*a bunching condition on the hyperbolicity exponents *(which always holds in dimension two - ok for 2d billiards!)*.

$\Rightarrow$ The results of BG2 cover those of Demers-Liverani and apply in particular to Lozi maps with an invariant domain.

To finish:

-a formal statement of the main result of BG2 (in the simplest case);

-a few words on the role of the bunching condition.

Theorem (BG2): Let $(O_i)_{i\in I}$ be finitely many pairwise disjoint open subsets covering Lebesgue almost all M, so that each ∂O_i is a finite union of C^1 hypersurfaces, and for each $i \in I$, let T_i be a C^2 injective local diffeomorphism, defined on a neighborhood of $\overline{O_i}$ in M. Assume that $T : M \to M$ satisfies $T|_{O_i} = T_i|_{O_i}$ and in addition:

(Hyperbolicity) There exist two continuous families of transverse cones $\mathcal{C}_i^{(u)}$ (d_u-dimensional), $\mathcal{C}_i^{(s)}$ (d_s-dimensional), such that:

$$DT_i(q)\mathcal{C}_i^{(u)}(q) \subset \mathcal{C}_j^{(u)}(T_i(q))\,,$$

there exists $\lambda_{i,u}(q) > 1$ such that $|DT_i(q)v| \geq \lambda_{i,u}(q)|v|$ on $\mathcal{C}_i^{(u)}(q)$; and

$$DT_i^{-1}(T_i(q))\mathcal{C}_j^{(s)}(T_i(q)) \subset \mathcal{C}_i^{(s)}(q)\,,$$

and there exists $\lambda_{i,s}(q) \in (0,1)$ such that $|DT_i^{-1}(T_i(q))v| \geq \lambda_{i,s}^{-1}(q)|v|$ on $\mathcal{C}_j^{(s)}(T_i(q))$.

(Transversality) Each ∂O_i is a finite union of C^1 hypersurfaces $K_{i,k}$ which are everywhere transverse to the stable cones (i.e. their tangent space contains a d_u-dimensional subspace that intersects the stable cone only at 0).

(Bunching) Denote by $\Lambda_{i,s}^{(n)}(q) \leq \lambda_{i,s}^{(n)}(q)$, $\Lambda_{i,u}^{(n)}(q) \geq \lambda_{i,u}^{(n)}(q)$ the strongest and weakest contraction/expansion coefficients of T_i^n at q. Assume that for some $n \geq 1$ and $\beta \in (0,1)$

$$\sup_{i\in I^n,\, q} \frac{\lambda_{i,s}^{(n)}(q)^{1-\beta}\Lambda_{i,u}^{(n)}(q)^{1+\beta}}{\lambda_{i,u}^{(n)}(q)} < 1\,.$$

Under the assumptions of the previous slide:

Let $1 < p < \infty$ and let $u, s \in \mathbb{R}$ be so that

$$1/p - 1 < s < 0 < u < 1/p\,, \quad -\beta < u + s < 0\,.$$

Then there exists a Banach space $\mathcal{B}$ of distributions on M, containing C^1, in which $L^\infty \cap \mathcal{B}$ is dense, and on which the operator defined on L^∞ by

$$\mathcal{L}\phi = \sum_i \chi_{T_i(\Omega_i)} \frac{\phi \circ T_i^{-1}}{|\det DT_i \circ T_i^{-1}|}$$

extends continuously, with essential spectral radius at most

$$\lim_{n\to\infty} (D_n^b)^{1/(pn)} \cdot (D_n^e)^{(1/n)(1-1/p)} \cdot \left\| |\det DT^n|^{1/p-1} \max(\lambda_{u,n}^{-u}, \lambda_{s,n}^{-(u+s)}) \right\|_{L^\infty}^{1/n} ,$$

where the hyperbolicity exponents are $\lambda_{s,n}(q) = \sup_i \lambda_{i,s}^{(n)}(q)\,, \quad \lambda_{u,n}(q) = \inf_i \lambda_{i,u}^{(n)}(q)$, and the complexity exponents are

$$D_n^b = \max_{q \in X_0} \mathrm{Card}\{i \in I^n \mid q \in \overline{O_i}\}\,, \quad D_n^e = \max_{q \in X_0} \mathrm{Card}\{i \in I^n \mid q \in \overline{T^n(O_i)}\}\,.$$

Comparing BG1 and BG2:

Similarities: Compact embedding properties and Lasota-Yorke inequalities, with basic Triebel spaces as a building block (Strichartz, complex interpolation)..

New challenge: There is no differentiable system of charts adapted to the (a priori measurable) stable foliation.

Remedy:Consider the family of *all* differentiable charts compatible with the stable cones/define the anisotropic norm by taking the supremum over this family of admissible charts.

Caveat: For this to work, it is necessary to check that the family of admissible charts is invariant under composition by the (inverse) dynamics.

Basically a graph transform argument (à la Hadamard-Perron), but the implementation is a bit intricate (need to reparametrize into standard form, to glue charts together to avoid exponential proliferation, to combine this with a zoom-type argument...)

Possible improvements (besides billiards project):

-Sharper upper bound, replacing the product of complexity by hyperbolicity by a «pressure of boundary»-type expression

-Remove bunching condition?