Dispersing billiards with cusps and tunnels

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work in progress with N. Chernov and D. Dolgopyat

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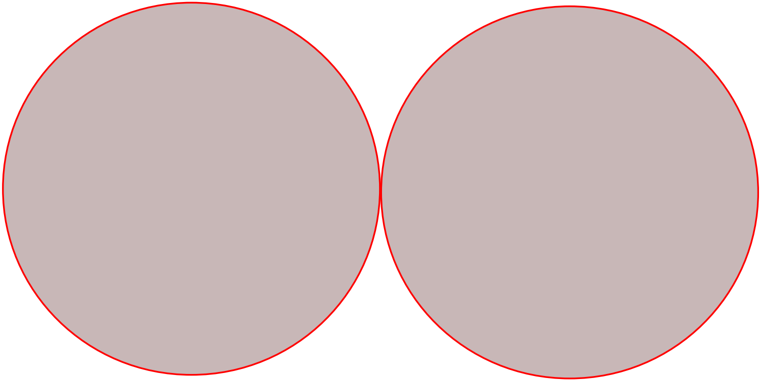
In a nutshell

- Billiards with cusps: slow decay of correlations, non-standard limit theorem;
- Billiards with tunnels: CLT, but variance blows up as $\varepsilon \to 0$.



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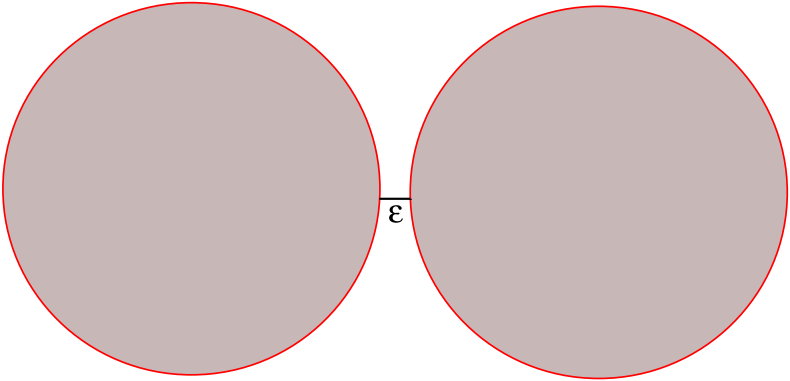
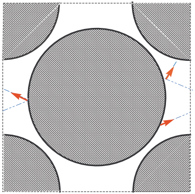


Outline

Billiards

$Q = \mathbb{T}^2 \setminus \bigcup_{k=1}^{K} C_k$ strictly convex scatterers

- Billiard flow : $S^t : \mathcal{M} \to \mathcal{M}$, $(q, v) \in \mathcal{M} = Q \times \mathbb{S}^1$, $|v| = 1$ Uniform motion within Q, elastic reflection at the boundaries
- Billiard map phase space: $M = \bigcup_{k=1}^{K} M_k$
- $(r, \phi) \in M_k$, r: arclength along ∂C_k, $\phi \in [-\pi/2, \pi/2]$ outgoing velocity angle
- invariant measure $d\mu = c \cos\phi \, dr \, d\phi$



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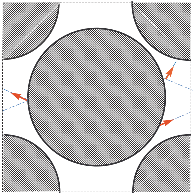

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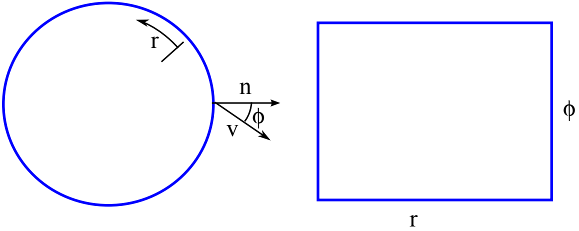

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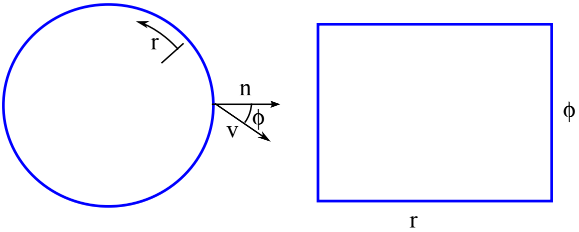

Sinai billiards

C_k are C^3 smooth and disjoint (no corner points);
finite horizon: flight length uniformly bounded from above

- Billiard map is ergodic, K-mixing (Sinai ’70)
- EDC: $f, g : M \to \mathbb{R}$ Hölder continuous, $\int f d\mu = \int g d\mu = 0$ let $C_n(f,g) = \mu(f \cdot g \circ T^n)$, then $|C_n(f,g)| \leq C\alpha^n$ for suitable $C > 0$ and $\alpha < 1$
  - Young ’98 – tower construction with exponential tails,
  - Chernov & Dolgopyat ’06 – standard pairs
- CLT: let $S_n f = f + f \circ T + ... + f \circ T^{n-1}$, then $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma)$ where $\sigma = \int f^2 d\mu + 2\sum_{n=1}^{\infty} C_n(f,f)$.
  Bunimovich & Sinai ’81, Chernov ’06, Melbourne ’06.
- Billiard flow: $F, G : \mathcal{M} \to \mathbb{R}$, $C_t(F,G)$:
  - stretched exponential bound, Chernov ’07 (approximate Markov partitions)
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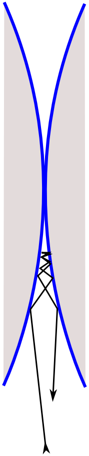
Cusp map



C_1 and C_2 touch tangentially – unbounded series of consecutive reflections in the vicinity of the cusp

- Reháček '95 ergodicity
- Machta '83 numerics and heuristic reasoning for $C_n(f,g) \asymp 1/n$
- Chernov & Markarian '07: $C_n(f,g) \leq C\frac{\log^2 n}{n}$
- Chernov & Zhang '08: $C_n(f,g) \leq C\frac{1}{n}$

Not summable $\Rightarrow$ non-standard limit law?

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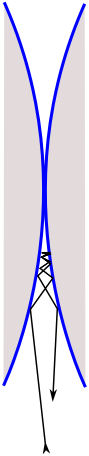

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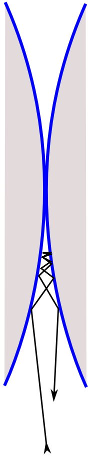

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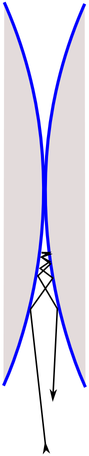

long collision series near the cusp correspond to bounded flow time – flow mixes faster?

Melbourne & B. ’08

- $C_t(F, G)$ decays faster than any polynomial
- S^t admits CLT (almost sure invariance principle)

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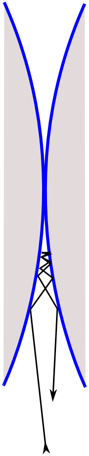

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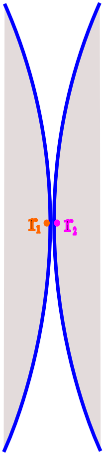
Cusp superdiffusion constant



“Result” (C)

- *Denote by $r_1 \in C_1$ and $r_2 \in C_2$ the two points that make the cusp.*
- *Let* $I_f = \int_{-\pi/2}^{\pi/2} (f(r_1, \phi) + f(r_2, \phi))\rho(\phi) d\phi$

  with $\rho(\phi) = \frac{\sqrt{\cos\phi}}{\int_{-\pi/2}^{\pi/2} \sqrt{\cos\phi} d\phi}$
- *if* $I_f \neq 0$ *then* $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$

  where $D_f = c^* I_f^2$ *and* c^* *is some numerical constant.*
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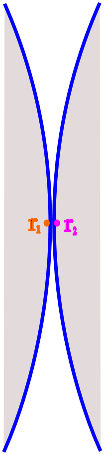

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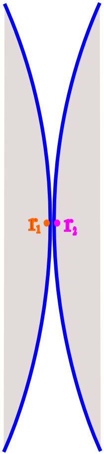

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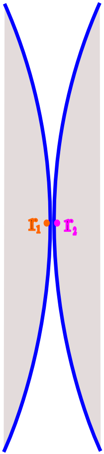

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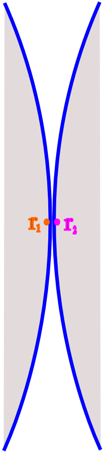
Remarks concerning the cusp flow



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$$g(x) = \int_0^{\tau(x)} G(x,t)dt,$$
- and we have $I_g = 0$ (as $\tau(x) = 0$ for $x = (r_1, \phi)$),
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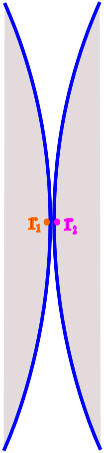

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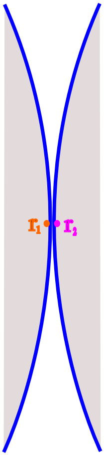

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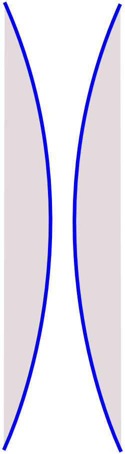
Blow-up of the variance in tunnels



"Result" (T)

Denote be $T_\varepsilon : M \to M$ the billiard map
same phase space, same $f : M \to \mathbb{R}$

- *for fixed $\varepsilon > 0$ this is a Sinai billiard, hence CLT:*
- $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_{f,\varepsilon})$ *with*
- $D_{f,\varepsilon} = D_f |\log \varepsilon| (1 + o(1))$

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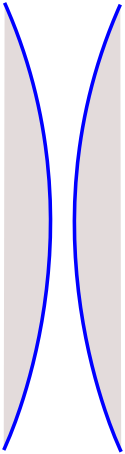

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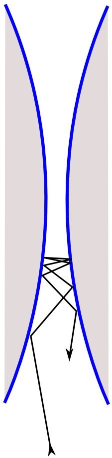

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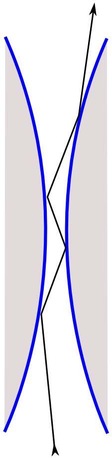

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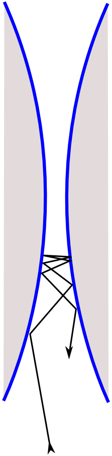

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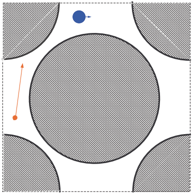
Motivation

1. Brownian Brownian motion – Chernov & Dolgopyat ’09



$m \ll M$ (separation of time scales)

SDE for large particle:

$dV = \sigma_Q(f)dW$

collisions of the heavy particle with the wall?

2. Triangular lattice with small opening

How does the planar diffusion depend on ε?

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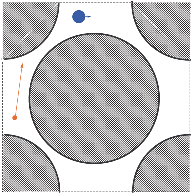

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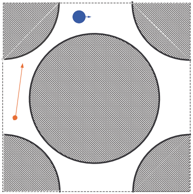

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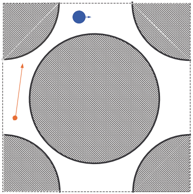

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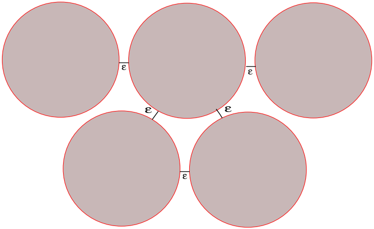
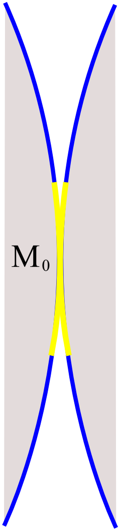


How does the planar diffusion depend on ε?

The first return map



Let $\hat{M} = M \setminus M_0$ where M_0 is a fixed small nbd. of the cusp.

- $\hat{T} : \hat{M} \to \hat{M}$ first return map
- $R : \hat{M} \to \mathbb{N}$ unbounded return time
- $\hat{f}(x) = \sum_{k=0}^{R(x)-1} f(T^k x)$ induced observable

limit law for $\hat{S}_n \hat{f}$ implies limit law for $S_n f$ (eg. Gouëzel '04)

$D_f = \mu(R) D_{\hat{f}} = \frac{D_{\hat{f}}}{\mu(\hat{M})}$

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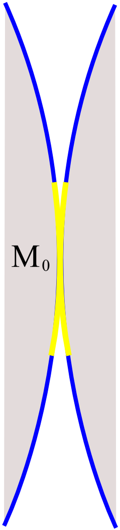

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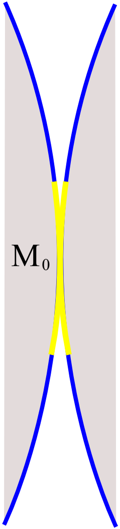

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Fast mixing of the first return map

Lemma (C1)

The map $\hat{T} : \hat{M} \to \hat{M}$ is uniformly hyperbolic and it satisfies the Growth Lemma ("Expansion prevails fractioning")

so that

- Young tower with exponential tails can be constructed
- standard pairs can be coupled at an exponential rate

Hence: EDC for Hölder observables

Lemma (C2)

$|\hat{\mu}(\hat{f} \cdot \hat{f} \circ \hat{T}^n)| \leq Ce^{-\alpha n}$ *with* $C > 0, \alpha < 1$ *for* $n \geq 1$
Not for $n = 0$ *as* $\hat{f}$ *is not Hölder and not in* L^2

Summarizing: the sequence $\hat{f} \circ \hat{T}^n$ behaves almost like an i.i.d. sequence

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Blow-up of $\hat{f}^2$

- $M_n = \{x \in \hat{M} | R(x) = n\}$ n-cell
- $L_n = \bigcup_{j \le n} M_j$ low cells, $H_n = \bigcup_{j > n} M_j$ high cells

Lemma (C3)

- $\hat{f}|_{M_n} = nI(1 + o(1))$

  (recall $I = c_1 \int_{-\pi/2}^{\pi/2} (f(r_1, \phi) + f(r_2, \phi))\sqrt{\cos(\phi)} d\phi$*)*
- $\hat{\mu}(H_n) = \frac{c_2}{n^2}(1 + o(1))$ *(here c_1, c_2 are numerical constants)*
- *hence* $\hat{\mu}(\hat{f}^2 \cdot \mathbf{1}|_{L_n}) = 2 \log n D_{\hat{f}}(1 + o(1))$

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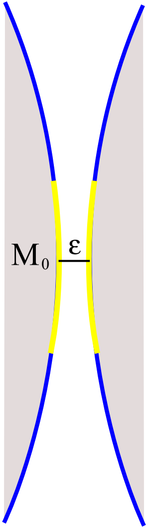
First return map for tunnel



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Return map $\hat{T}_\varepsilon : \hat{M} \to \hat{M}$ and return time R_ε depend on ε

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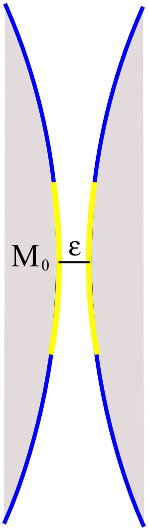

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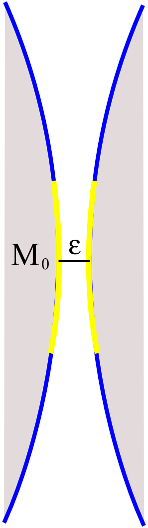

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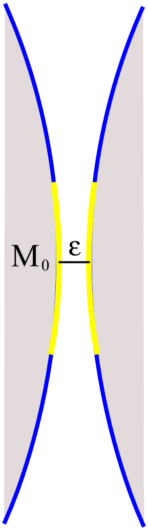

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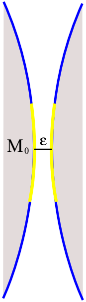

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$$\hat{\mu}(\hat{f}_\varepsilon^2) = |\log \varepsilon| D_{\hat{f}}(1 + o(1))$$

All these Lemmas require: detailed geometric analysis of the cells M_k (measures, unstable and stable dimensions etc...)

- For cusp, mostly (but not completely) done by Chernov & Markarian
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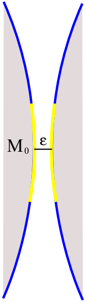

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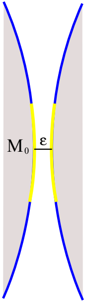

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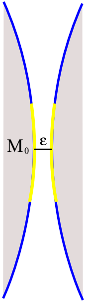

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Corner series

For simplicity assume that C_1 and C_2 are circles of radius 1.

Coordinates: α distance from cusp, $\gamma = \frac{\pi}{2} - \phi$

- while going down the cusp: α decreases, $\gamma : 0 \longrightarrow \frac{\pi}{2}$
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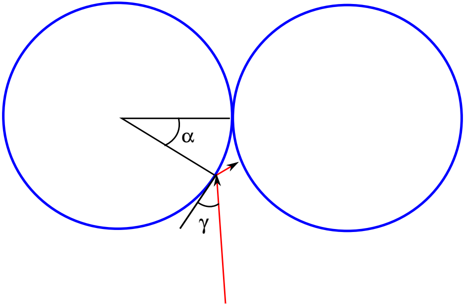

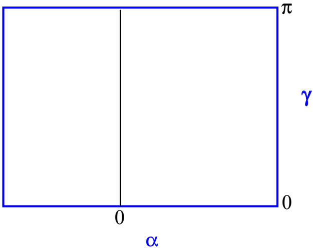

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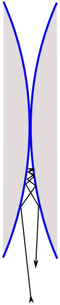

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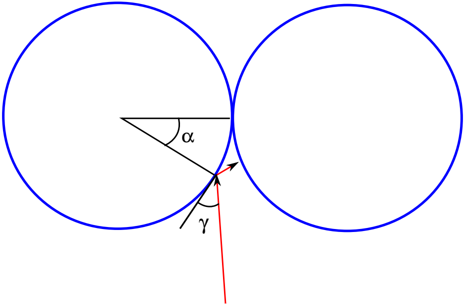
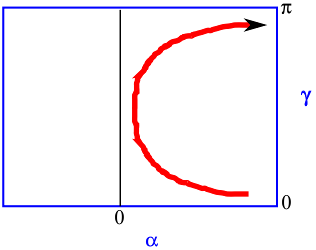
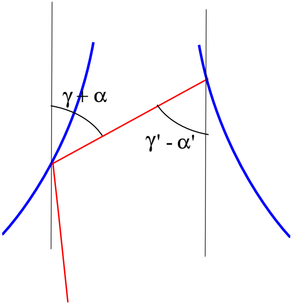




Equations of motion



$$\gamma' - \alpha' = \alpha + \gamma$$

$b = \sin\alpha - \sin\alpha'$;
$a = 2 - \cos\alpha - \cos\alpha'$
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$\sin\alpha' - \sin\alpha = -\frac{2-\cos\alpha'-\cos\alpha}{\tan(\alpha+\gamma)}$

- Throughout the corner series: $\alpha \ll 1$, $\alpha < \gamma$;
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$$\gamma' - \gamma \approx 2\alpha; \qquad \alpha' - \alpha \approx -\frac{\alpha^2}{\tan(\gamma)}.$$

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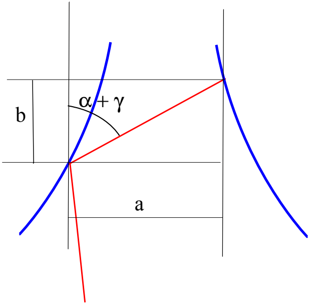

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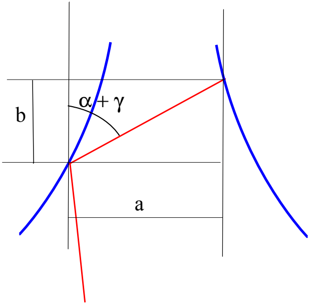

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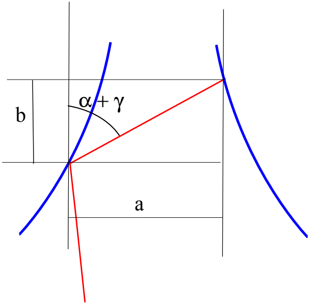

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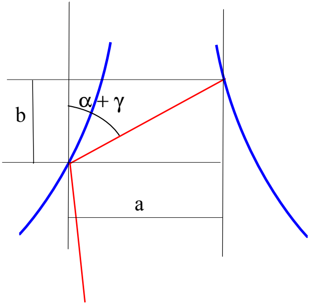

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Coordinates: α, γ as for cusp

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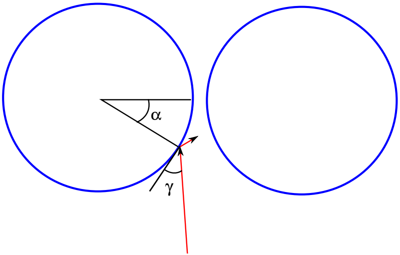

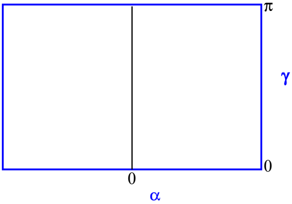

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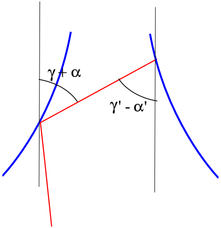

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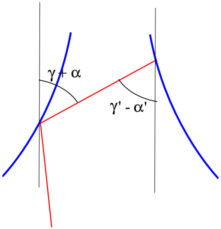

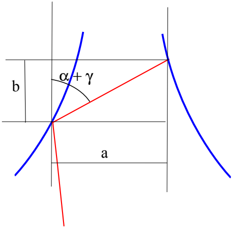

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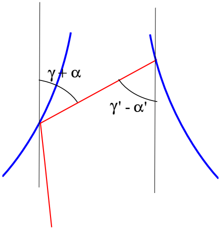

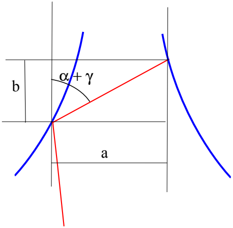

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Contribution to the variance: $\hat{\mu}(\hat{f}^2 \cdot \mathbf{1}_{L_{\frac{c}{\sqrt{\varepsilon}}}}) = D_{\hat{f}} |\log \varepsilon|$

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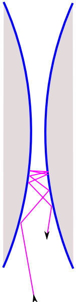

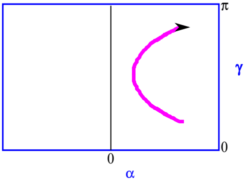

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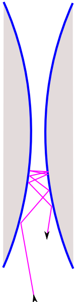

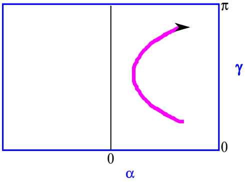

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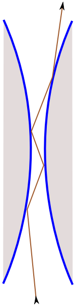

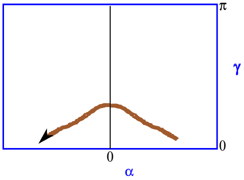

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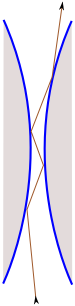

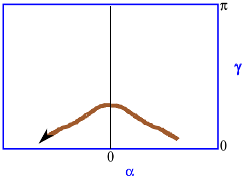

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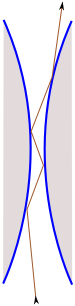

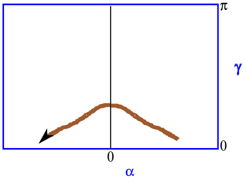

The third case

What is in between?

$\alpha = 0, \gamma = \pi/2$ is a hyperbolic fixed point (period two orbit)

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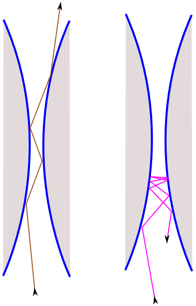

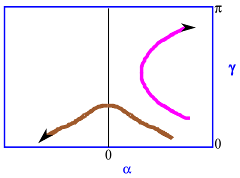

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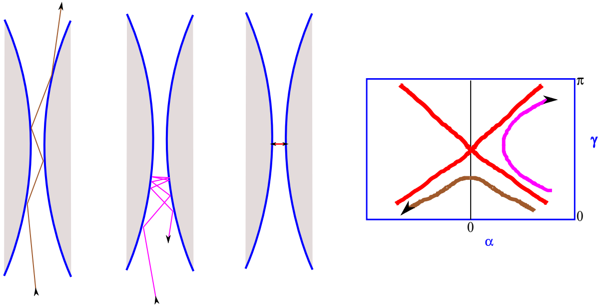

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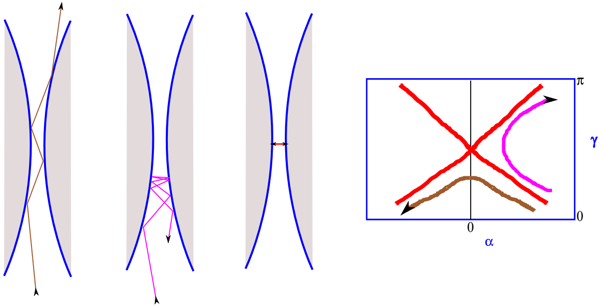

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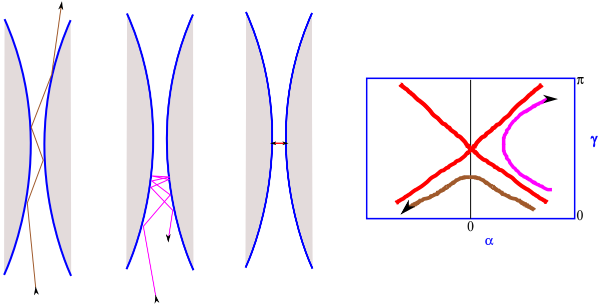

Summary and comparisions

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- Tunnel: $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_{f,\varepsilon})$ with $D_{f,\varepsilon} = |\log \varepsilon| D_f (1 + o(1))$

Related models:

1. Infinite horizon Lorentz gas and field of strength ε
2. Stadia
   - short-range correlations
   - what is ε?

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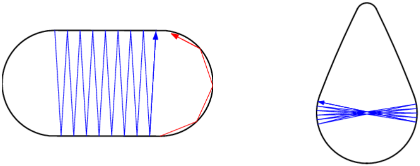
1. Infinite horizon Lorentz gas and field of strength ε
2. Stadia
    - short-range correlations
    - what is ε?



Summary and comparisions

- Cusp: $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$ with explicit D_f
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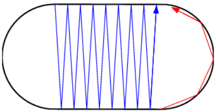

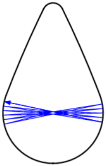

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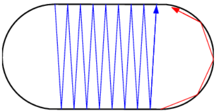
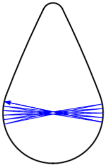
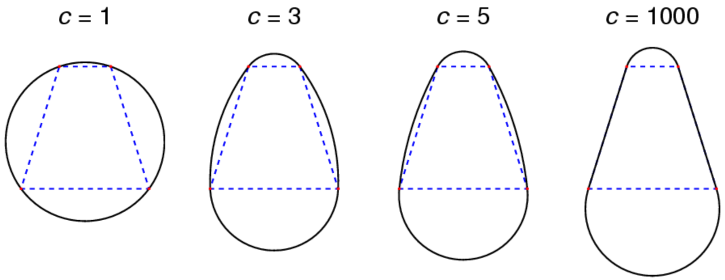




Generalized squashes



Numerics and heuristic reasoning:
Ergodicity for large enough finite c
(Halász, Sanders, Tahuilán, B.)

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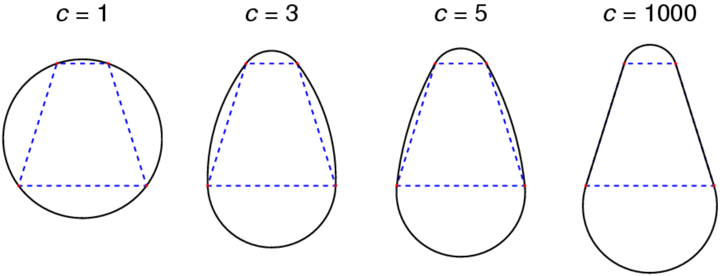

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