A canonical thickening of $\mathbb{Q}$ and the dynamics of continued fractions

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Credits

Joint work with Giulio Tiozzo (Harvard PhD student)
arXiv:1004.3790v1 [math.DS]

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C.C., S.Marmi, A.Profeti, G.Tiozzo: The entropy of *alpha*-contined fractions: numerical results arXiv:0912.2379v1 [math.DS]

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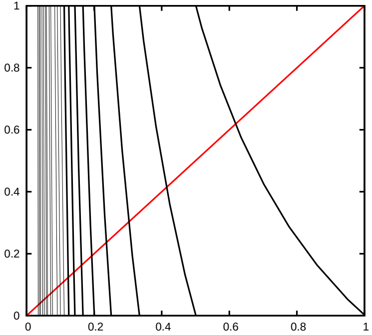

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The maps T_α

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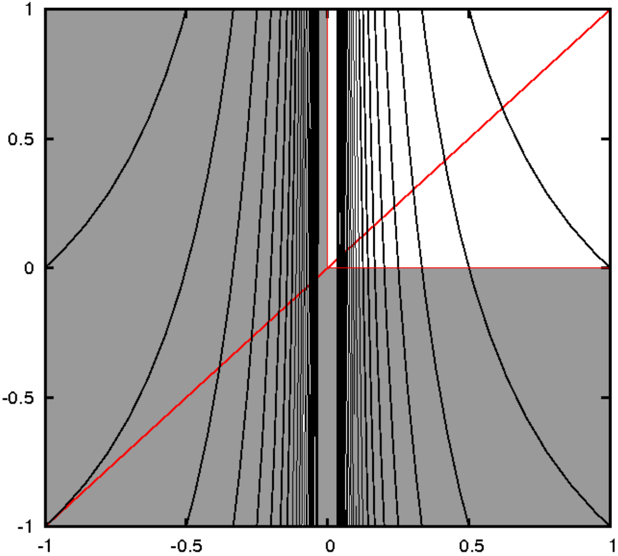

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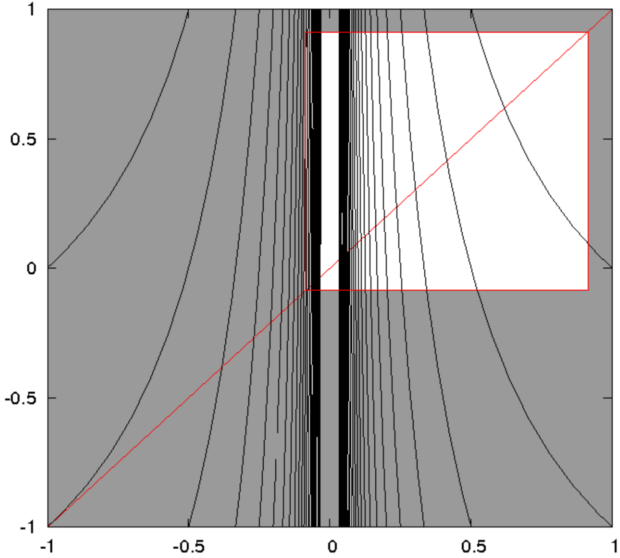

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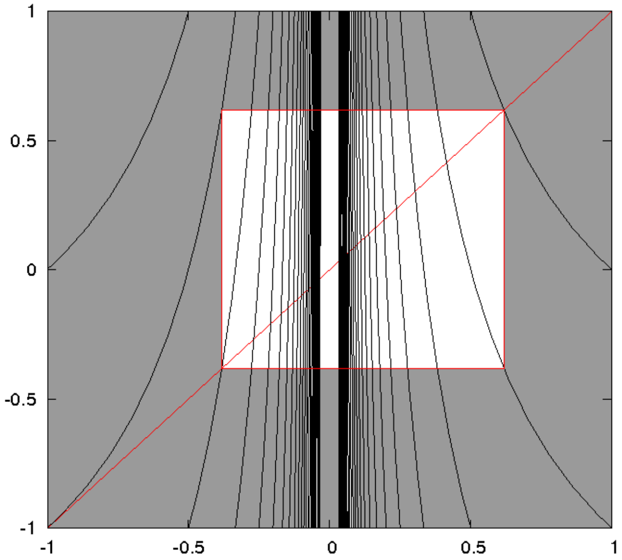
$T\alpha$ for $\alpha = (\sqrt{5} - 1)/2$



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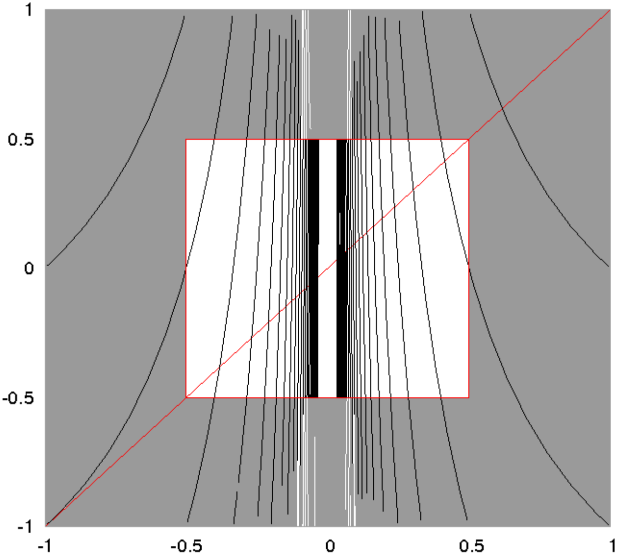

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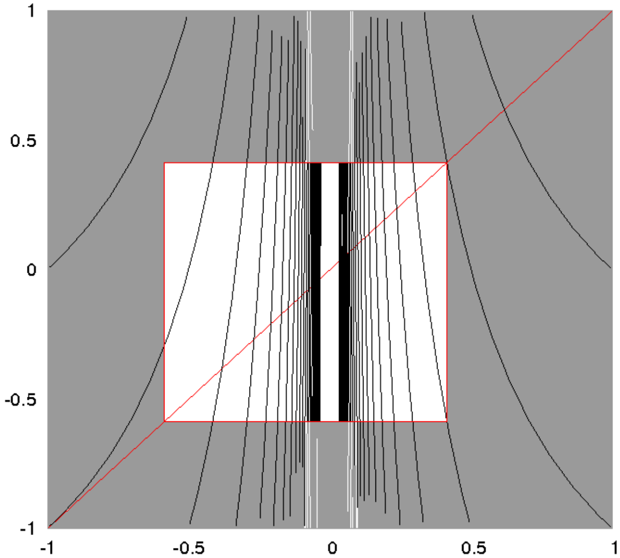

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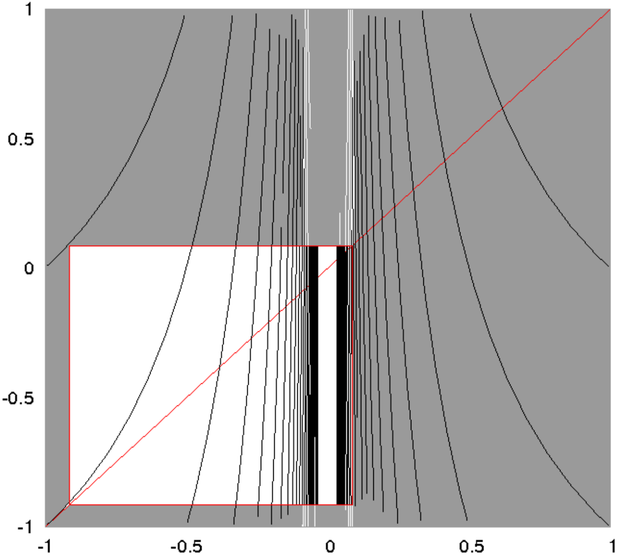

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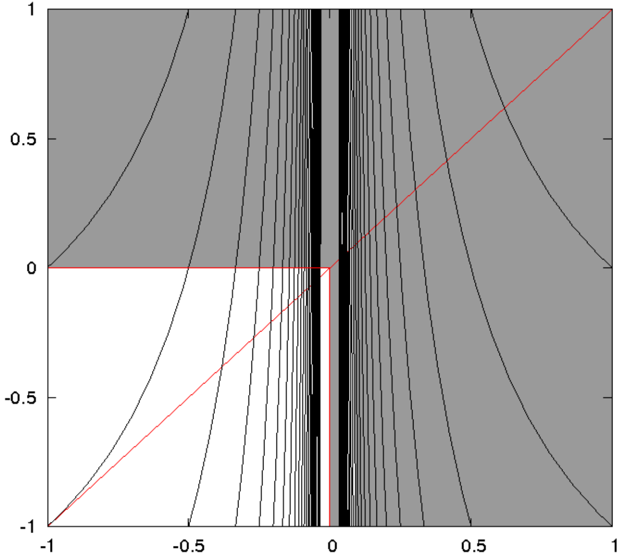

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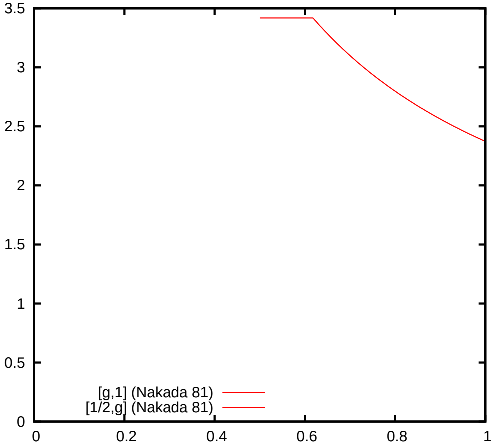
An historical account



H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. 4 (1981), 399-426

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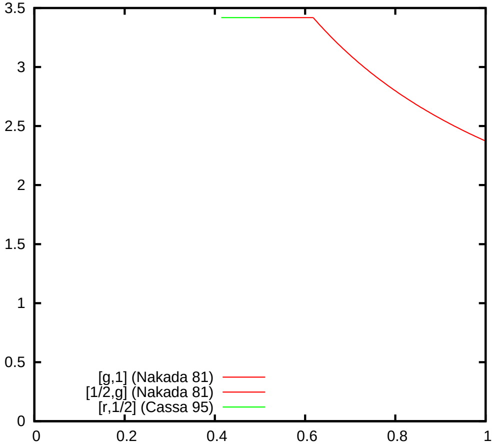

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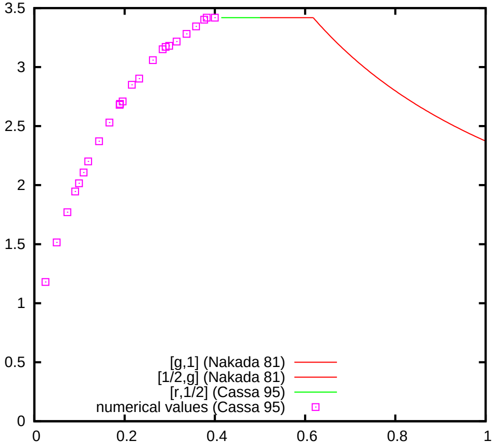

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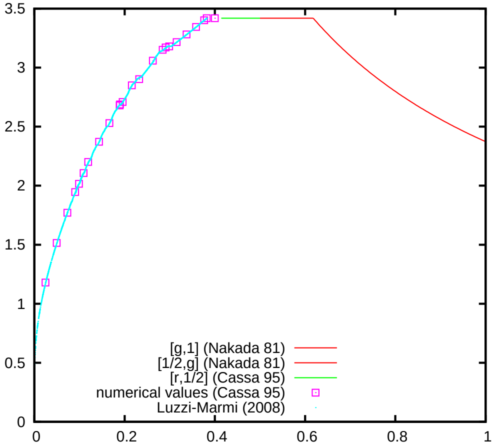
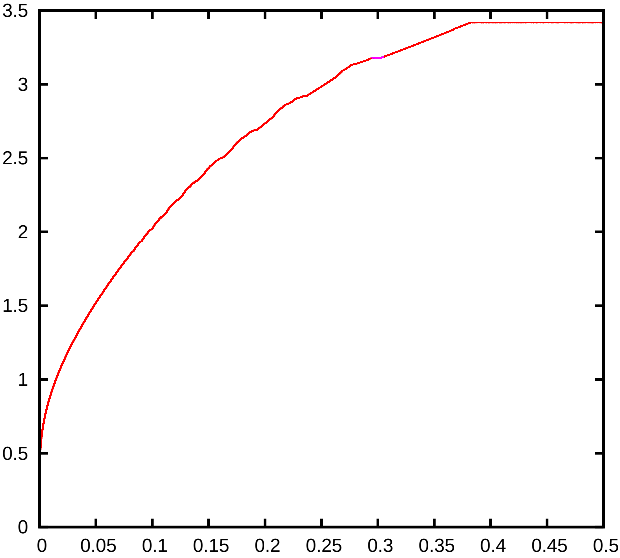


L. Luzzi, S. Marmi, *On the entropy of Japanese continued fractions*, Discrete and continuous dynamical systems, 20 (2008), 673-711.

Zooming in



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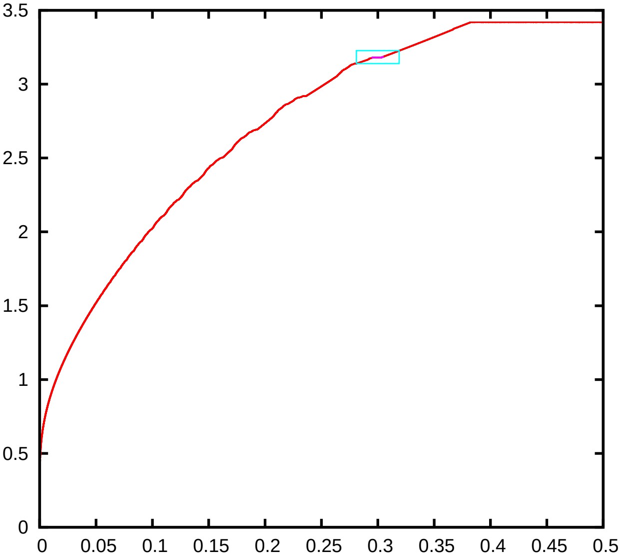

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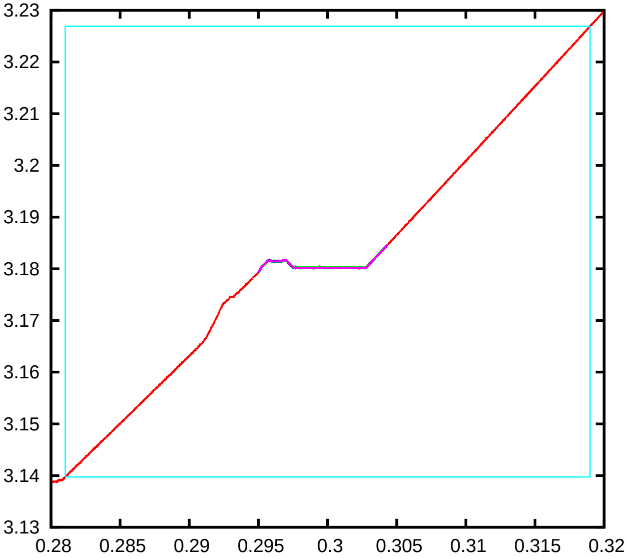

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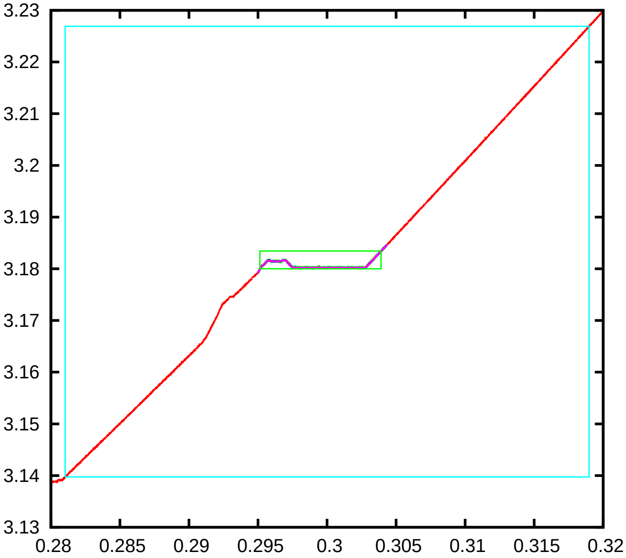
Zooming in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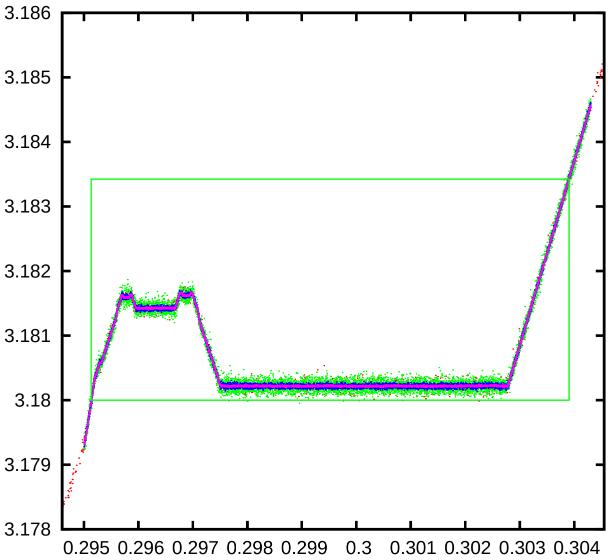
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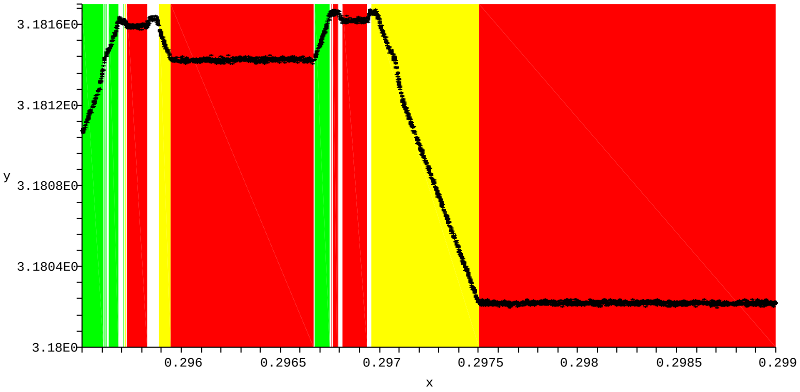

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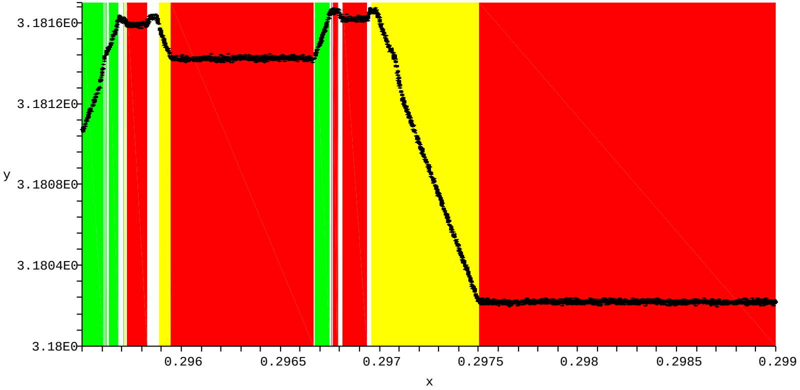

[NN] H. Nakada, R. Natsui, *The non-monotonicity of the entropy of α-continued fraction transformations*, Nonlinearity **21** (2008), 1207-1225

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- each of the cases (i), (ii) and (iii) takes place at least on one infinite family of disjoint matching intervals clustering at the origin ([NN], Thm. 3);

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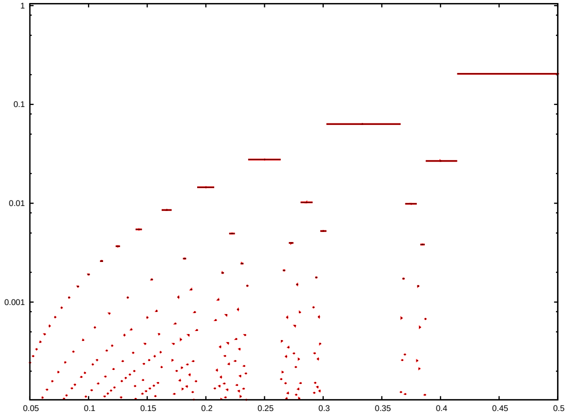
- **conjecture:** the union of all matching intervals is a **dense**, open subset of $[0, 1]$ with **full Lebesgue measure**.

Intervals of monotonicity of $\alpha \mapsto h(T_\alpha)$



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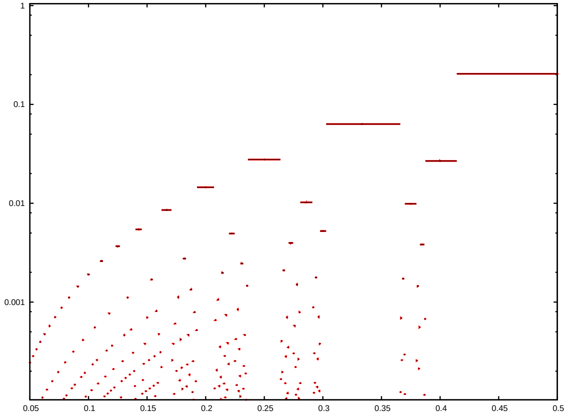

C.C., S.Marmi, A.Profeti, G.Tiozzo: The entropy of *alpha*-contined fractions: numerical results
arXiv:0912.2379v1 [math.DS]

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The value r will be called the *pseudocenter* of the interval (a, b).

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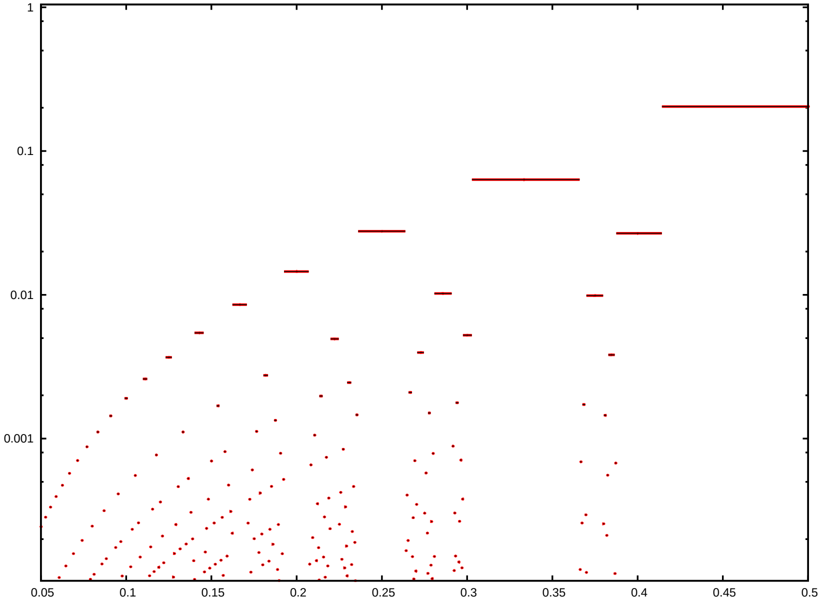

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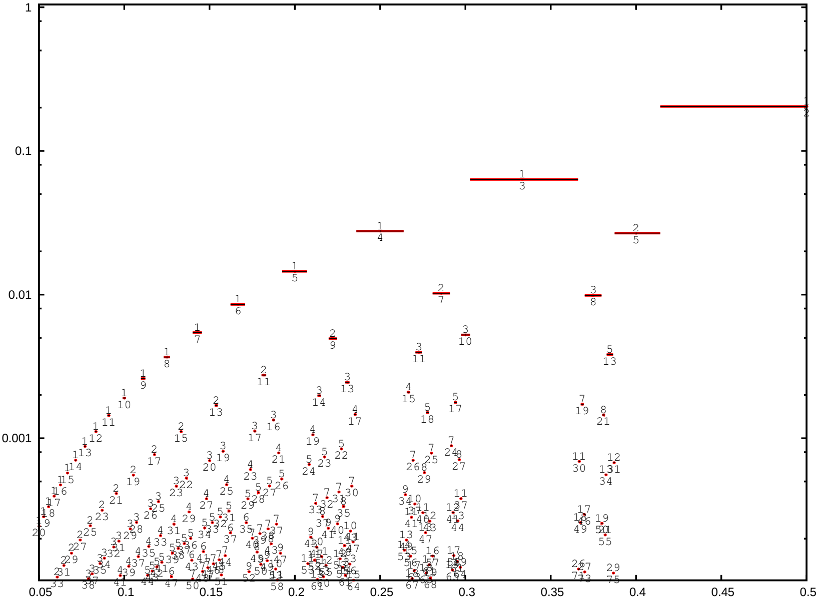

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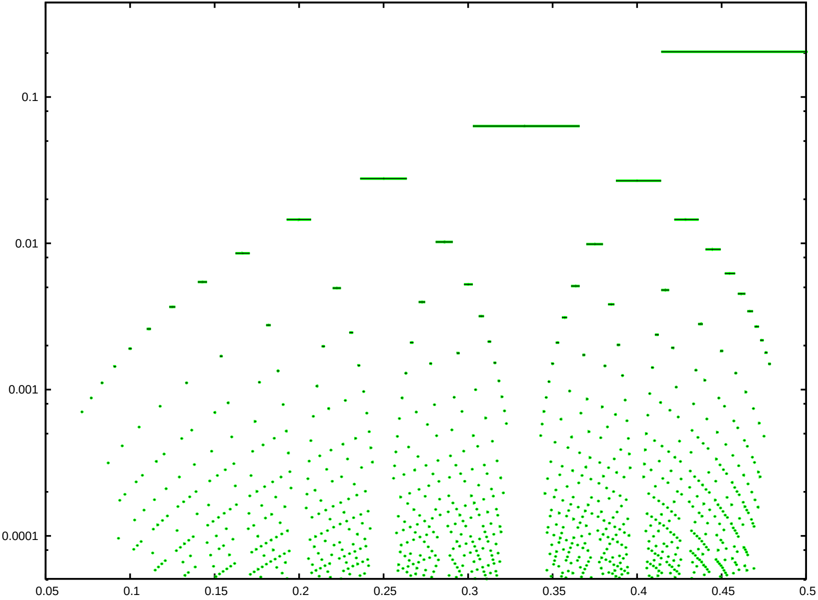

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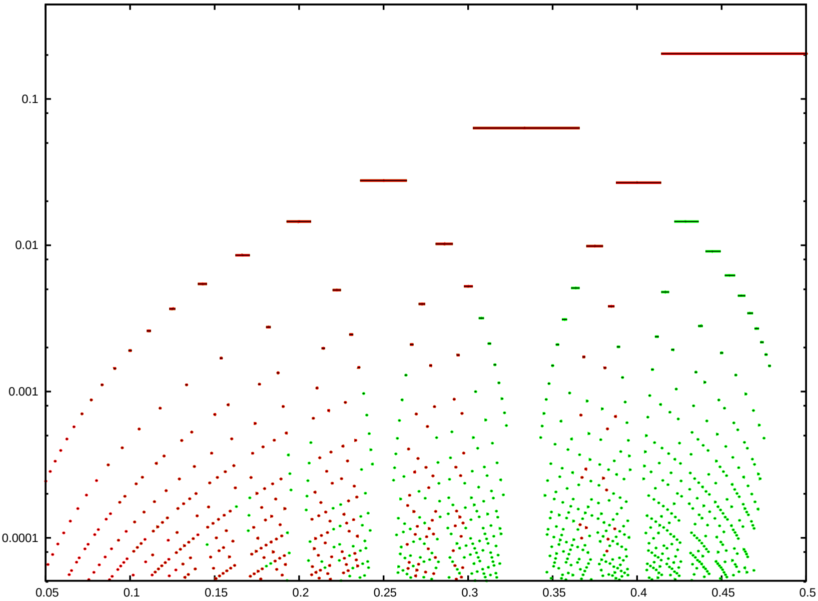

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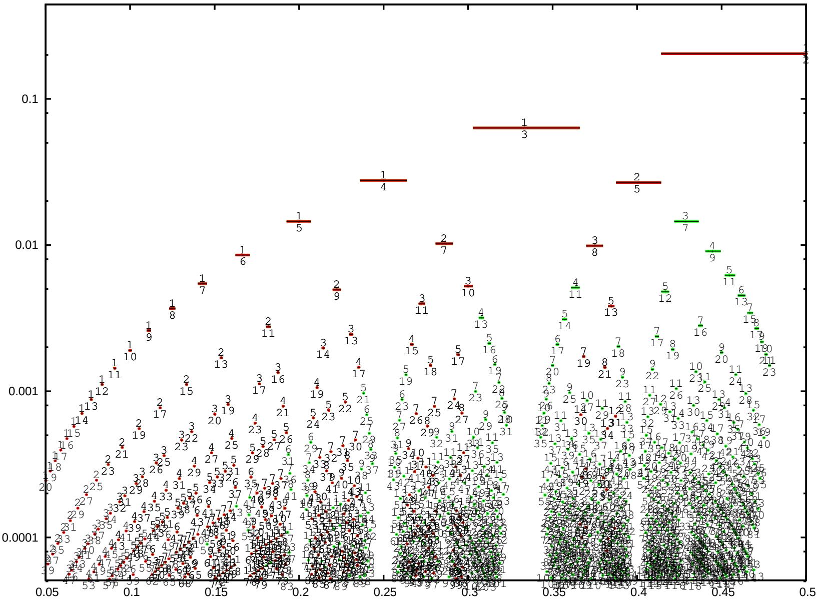

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In particular, distinct maximal intervals do not intersect.

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The exact same definition also gives a total ordering on the space of infinite strings.

If S and T have the same length (finite or infinite),

$$S < T \iff [0; S] < [0; T]$$

i.e. this order is just the pull back the order structure on $\mathbb{R}$, via identification of a string with the value of the corresponding c.f.

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In the first case, all cyclic permutations produce bigger values; in the second case, the cyclic permutation that carries a_k in the first place produces a value smaller than the original one.

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Remark: each maximal interval is almost completly covered by the matching ing intervals of NN (what is not covered is a closed countable set)

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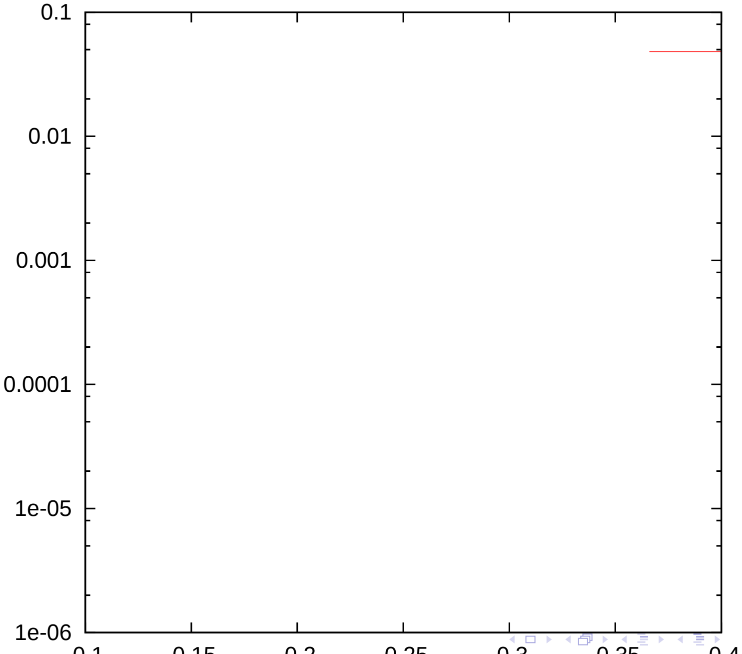
The end

Taking off quadratic intervals (via bisection)



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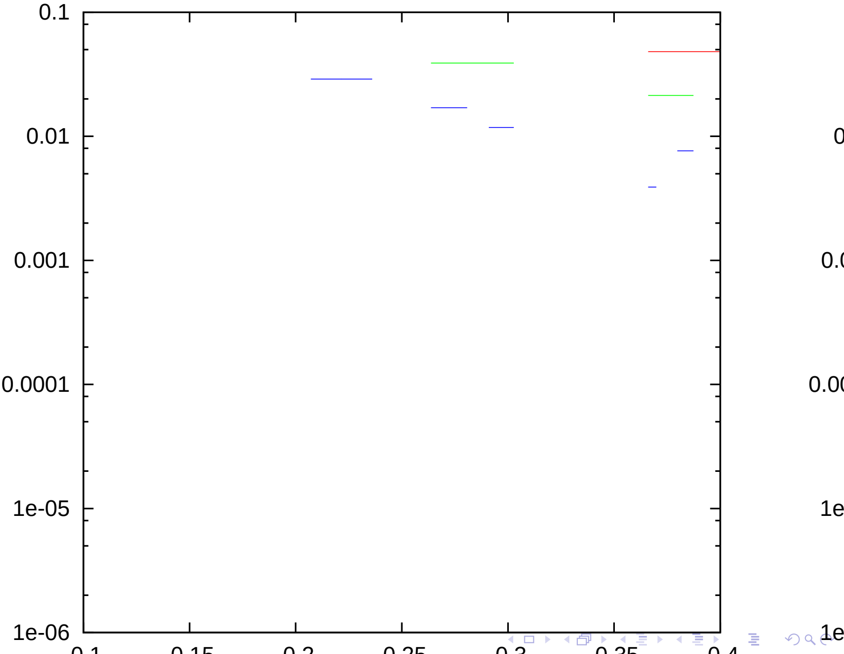

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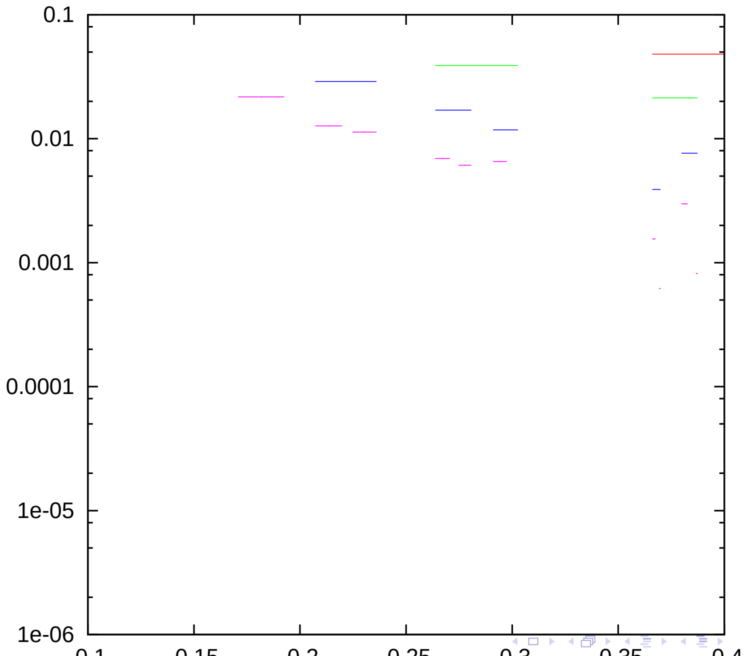

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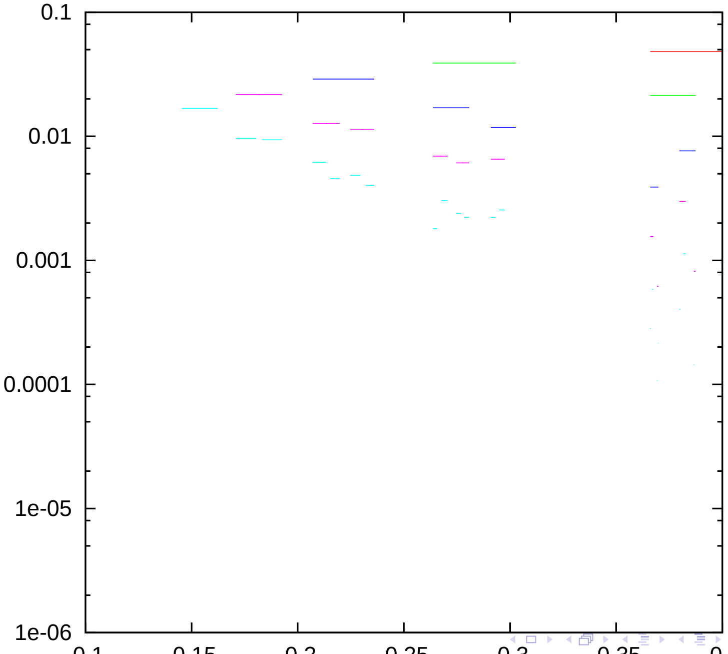

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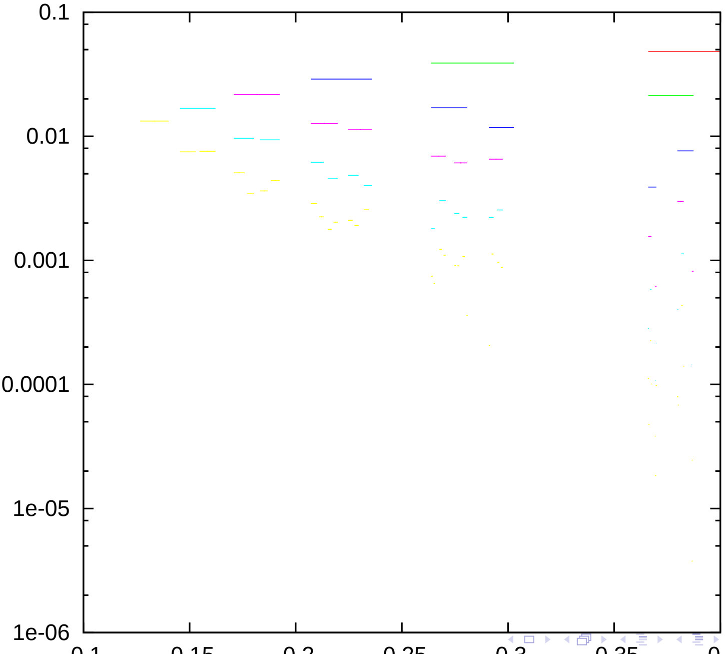

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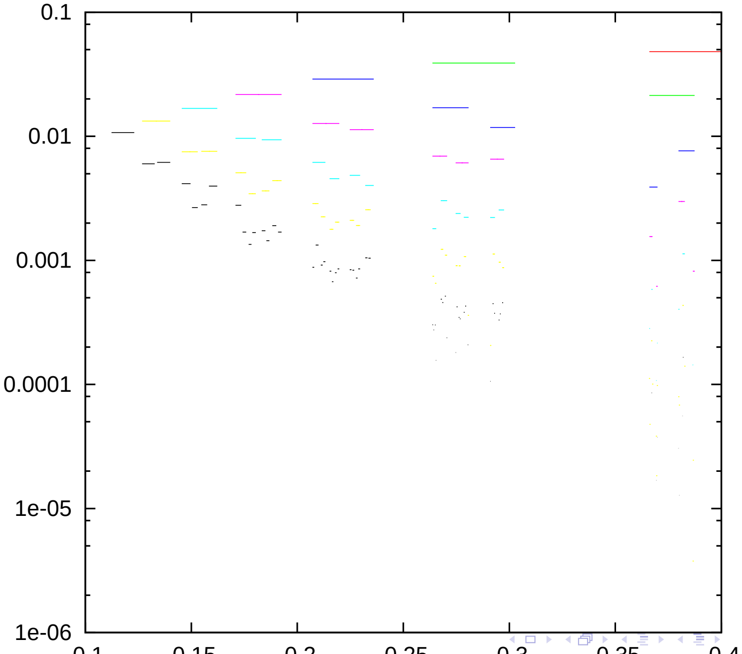

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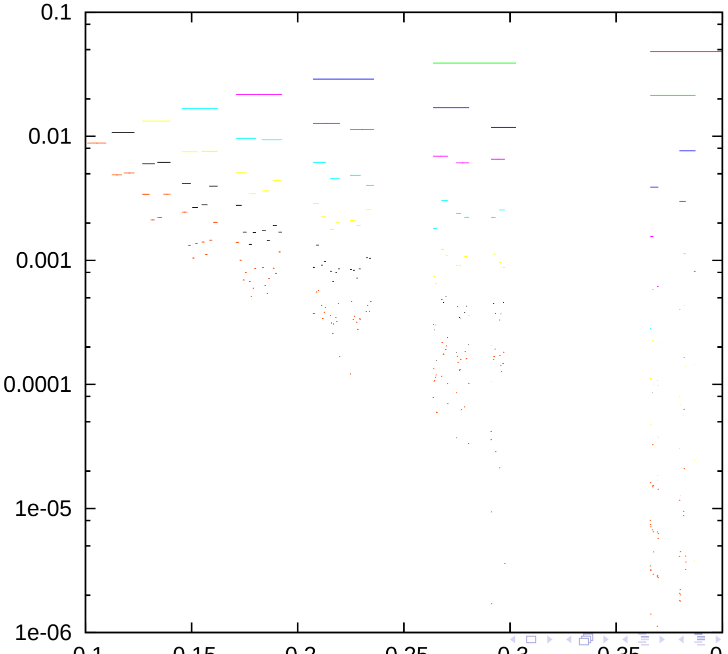

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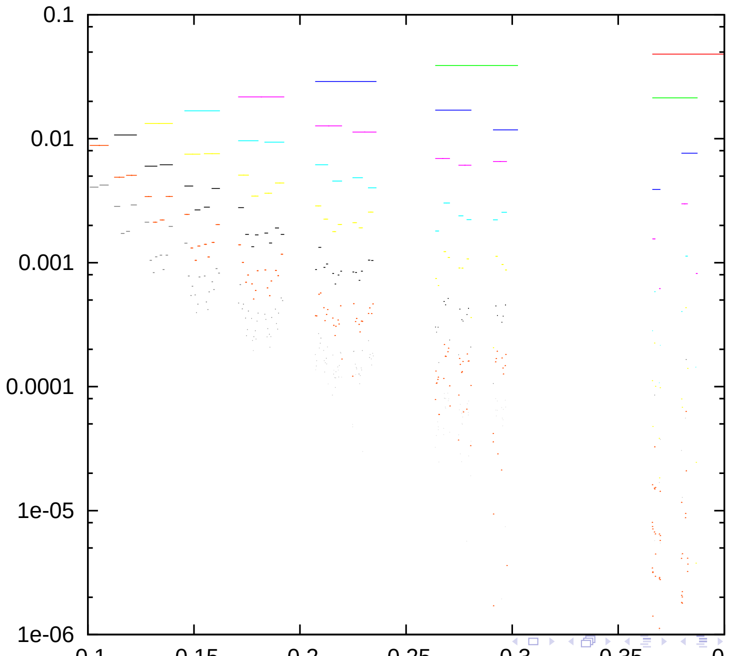

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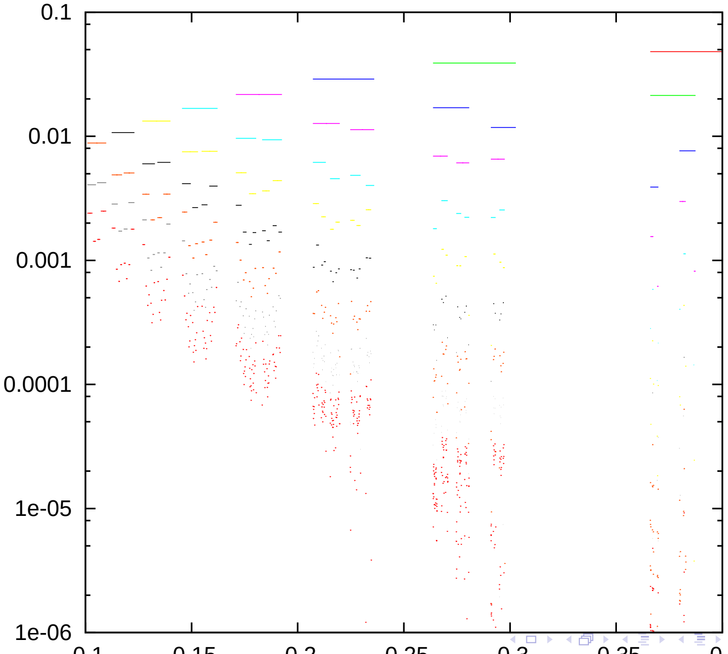

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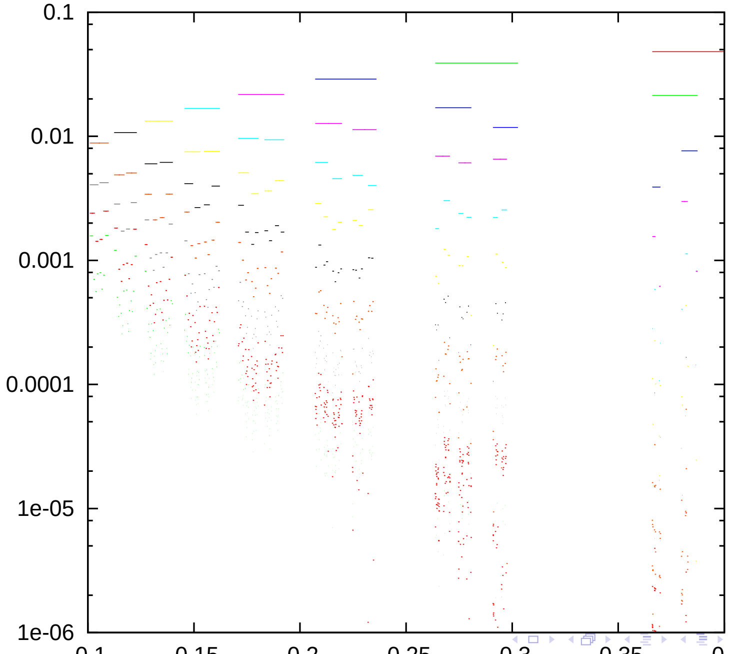

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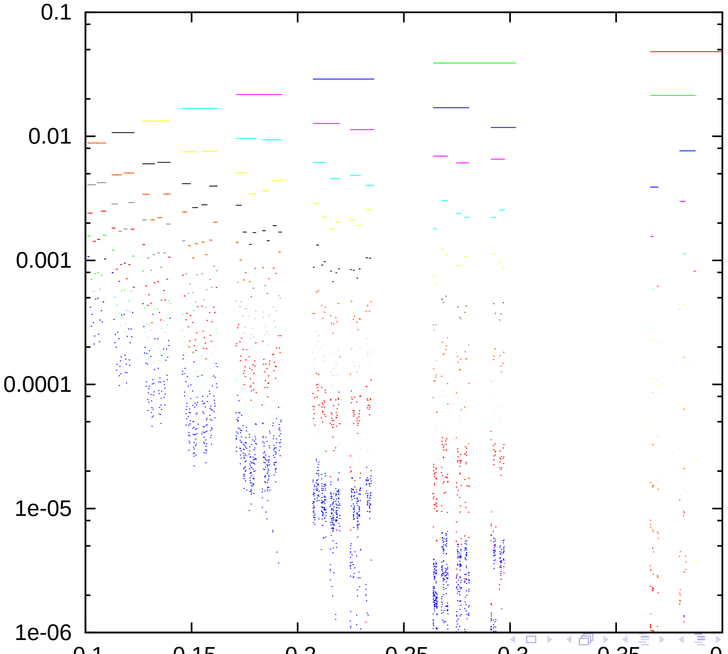
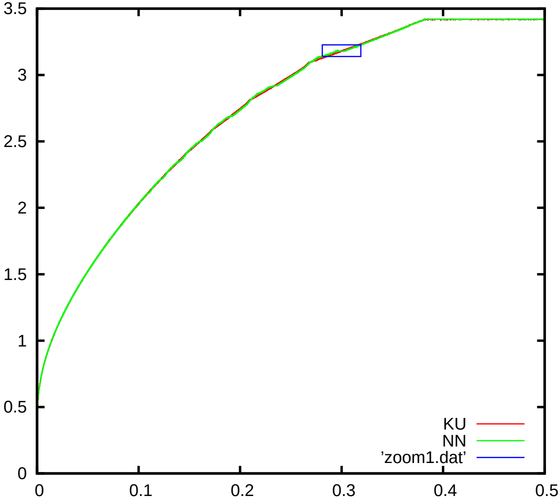


The effect of flipping: entropy



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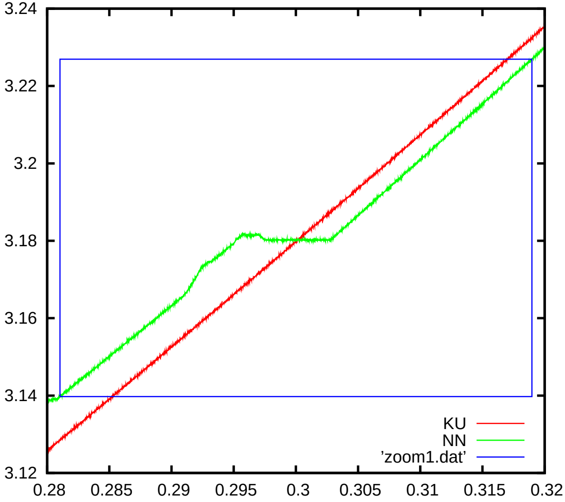
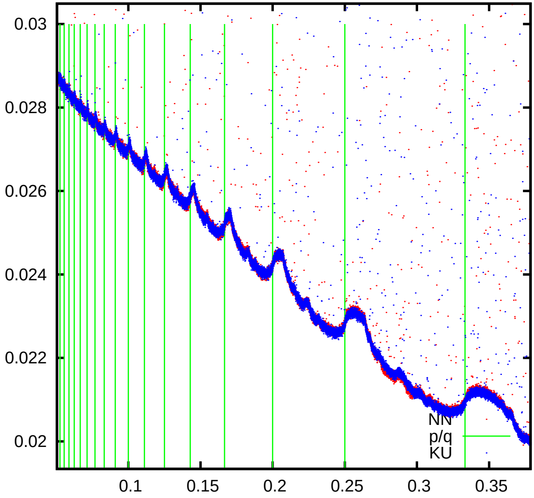


The effect of flipping: variance



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Yes
[NN]

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Probably not [CMPT].

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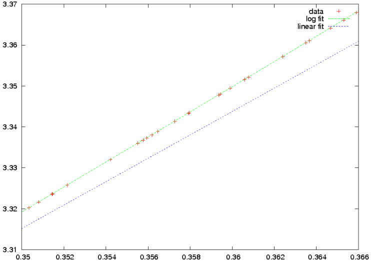

Some more questions

What about the self similar structure? how is it generated?

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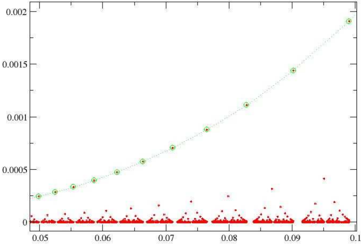

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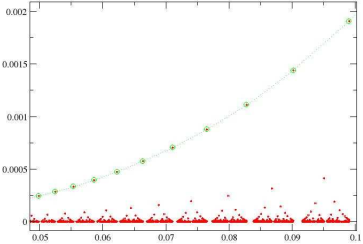

See [CT]

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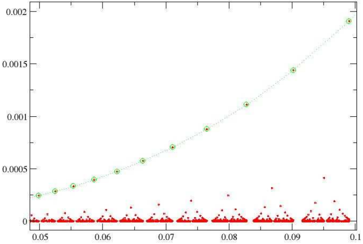

See [CT]

What about natural extensions?

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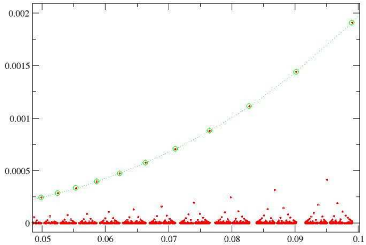

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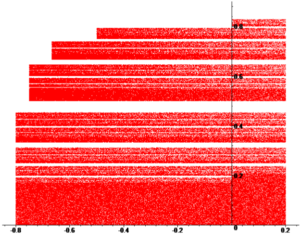

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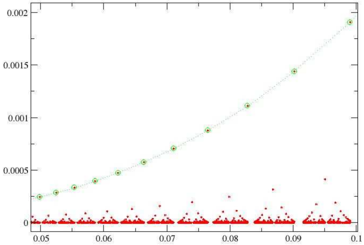

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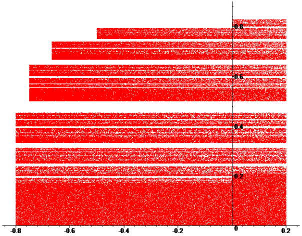

Who knows!?

The end