On the Limit Curlicue Process for Theta Sums

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Joint work with Jens Marklof, University of Bristol.

Theta Sums and Curves

- Let $\alpha, \beta \in \mathbb{R}$ and $N \in \mathbb{N}$. Let $e(z) := e^{2\pi i z}$. Define the **theta sum**

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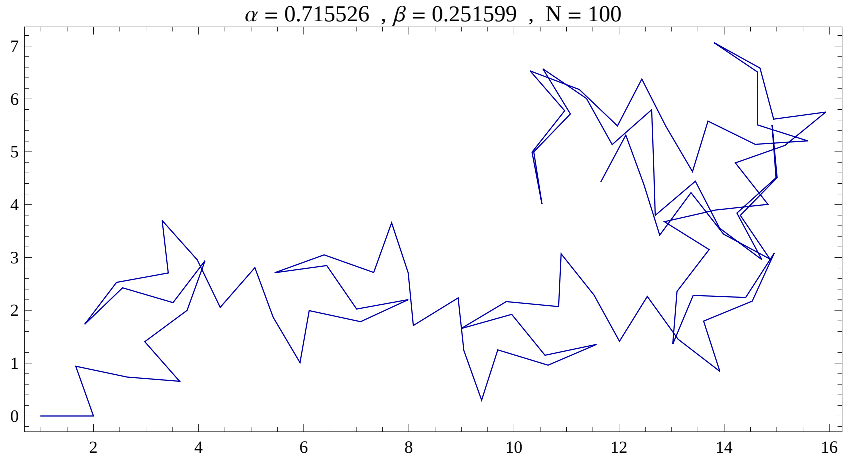
In particular $S_{1,N}(\alpha, \beta) = S_N(\alpha, \beta)$.
- We study the **curve** $[0, 1] \to \mathbb{C}$ obtained by linearly interpolating the N complex points $S_{\frac{1}{N},N}(\alpha, \beta), S_{\frac{2}{N},N}(\alpha, \beta), \ldots, S_{\frac{N-1}{N},N}(\alpha, \beta), S_{1,N}(\alpha, \beta)$.

Example (1)

$$\{S_{t,N}(\alpha,\beta)\}_{t\in[0,1]}$$



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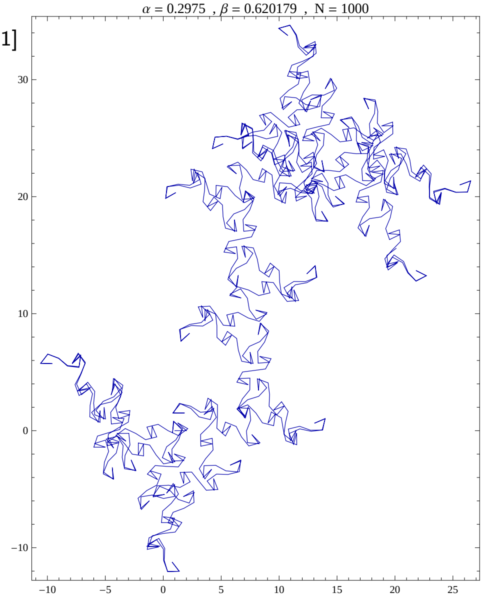

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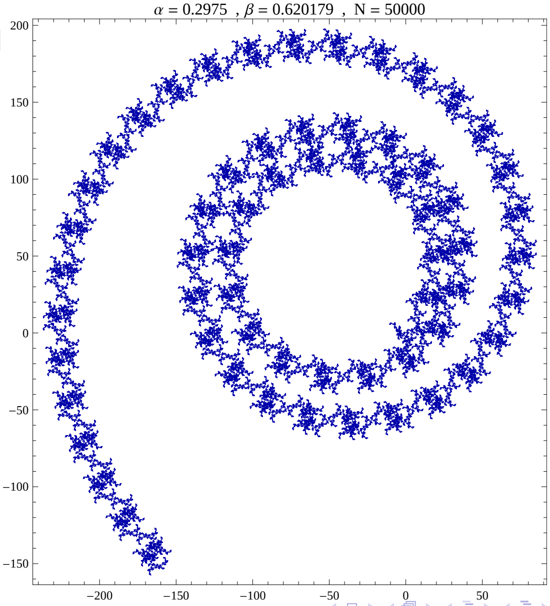

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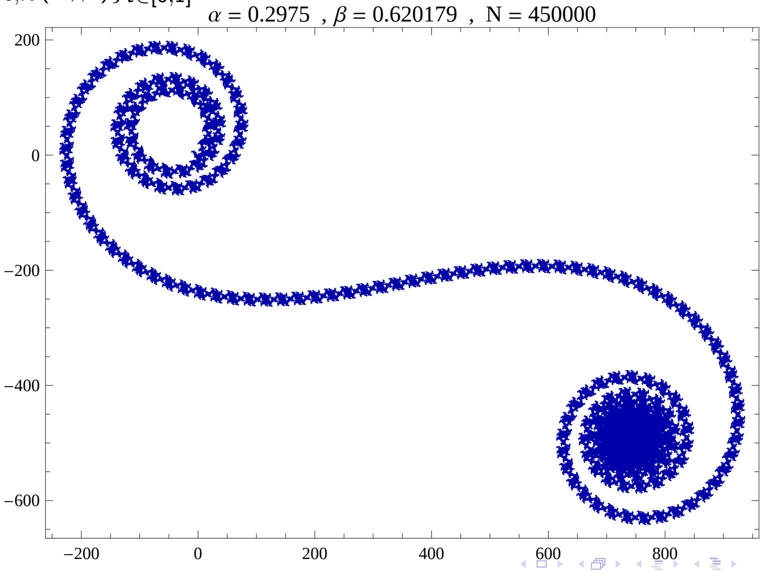
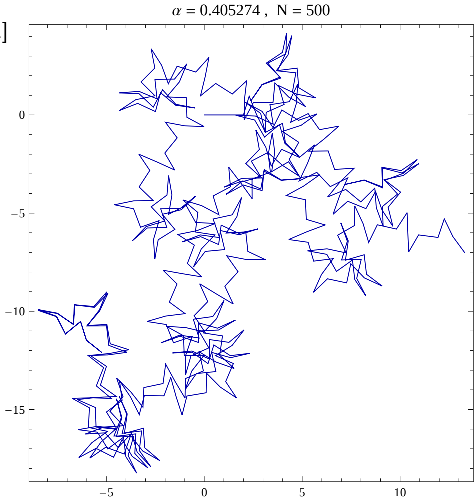


Example (3)

$\{S_{t,N}(\alpha, 0)\}_{t\in[0,1]}$



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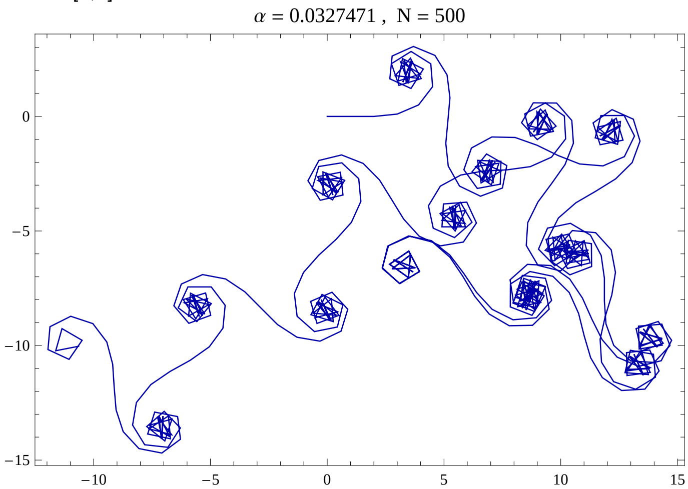

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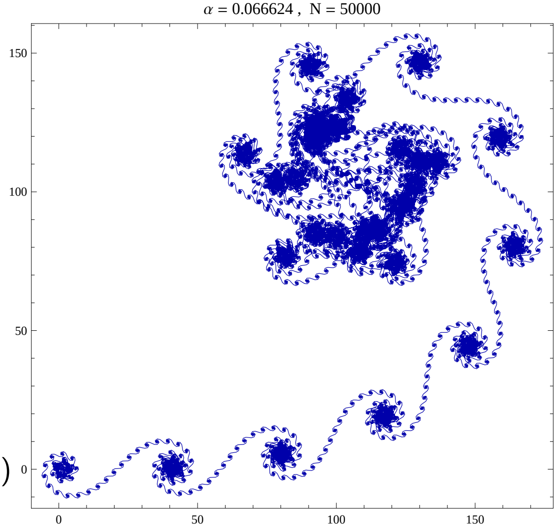

"curlicues"
(Berry-Goldberg 1988)

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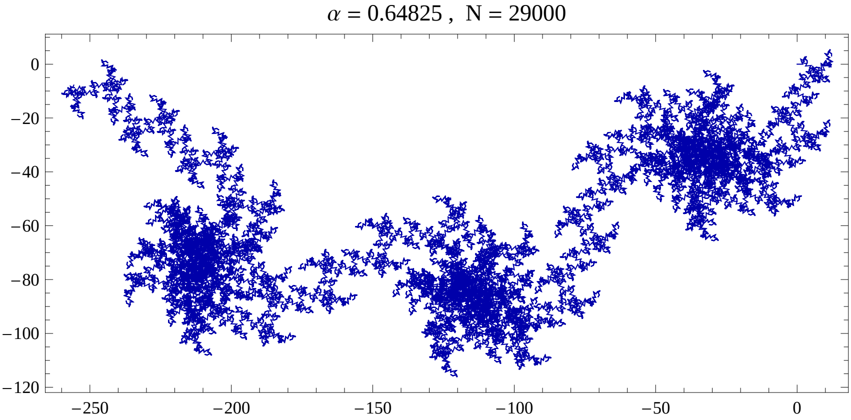

"Misshapen chaos of well-seeming forms"
(Romeo, *Romeo and Juliet*)

Growth of $|S_{1,N}(\alpha,\beta)|$

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- **Theorem B** (Friedler-Jurkat-Körner 1977; Flaminio-Forni 2006). For every increasing function $b:(1,\infty)\to(0,\infty)$ s.t. $\int_1^\infty u^{-1}b^{-4}(u)\mathrm{d}u<\infty$, there exists a full measure set $\mathcal{G}_b$ such that for every $\alpha\in\mathcal{G}_b$ and every $\beta\in\mathbb{R}$ the following holds:

$$|S_{1,N}(\alpha,\beta)|\leq C\sqrt{N}\,b(N).$$

Limiting Distributions for $|S_{1,N}(\alpha,\beta)|$ and $S_{1,N}(\alpha,\beta)$

- **Theorem C** (Jurkat-Van Horne 1981-3). There exists a function Ψ such that for all (except for countably many) $a, b \in \mathbb{R}_{\geq 0}$

$$Leb\left(\left\{\alpha \in [0,1]:\ a < N^{-\frac{1}{2}}|S_{1,N}(\alpha,0)| < b\right\}\right) \stackrel{N\to\infty}{\longrightarrow} \Psi_0(a,b).$$

The distribution function Ψ_0 is not Gaussian (it has only finitely many moments: $\Psi_0(R,\infty) \leq \frac{C_0}{R^4}$).

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- *Marklof's Thesis*: For $\beta \notin \mathbb{Q}$ the analogous statement for $S_{1,N}(\alpha,\beta)$ is true but $\mathrm{P}_\beta = \mathrm{P} \neq \mathrm{P}_0$ does not depend on β and has slightly better decay: $\Psi_\beta(R,\infty) \leq \frac{C}{R^6}$ (and does not depend on β).

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- **Theorem 1** (C. 2009, to appear in *Ann.Inst. Henri Poincaré*). Fix $0 \leq t_1 < t_2 < \ldots < t_k \leq 1$. There exists a probability measure $\mathrm{P}^{(k)}_{0;t_1,\ldots,t_k}$ (**curlicue measure**) on $\mathbb{C}^k$ such that for all *nice*, open $A \subset \mathbb{C}^k$

$$\lambda\left(\left\{\alpha \in [0,1] : \left(\gamma_N^{\alpha,0}(t_j)\right)_{j=1}^{k} \in A\right\}\right) \overset{N\to\infty}{\longrightarrow} \mathrm{P}^{(k)}_{0;t_1,\ldots,t_k}(A),$$

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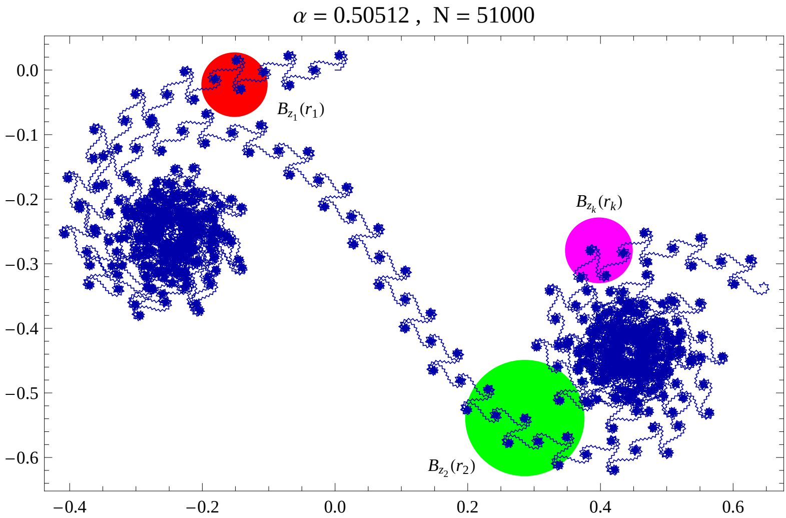
- **Thm D** corresponds to $\lambda = \mathrm{Leb}$, $k = 1$ and $t_1 = 1$. In particular $\mathrm{P}_0 = \mathrm{P}^{(1)}_{0;1}$.



$\alpha = 0.50512$, N = 51000
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
$B_{z_1}(r_1)$
$B_{z_k}(r_k)$
$B_{z_2}(r_2)$

In other words:

- For every N, given the map $([0,1],\mathcal{B},\lambda)\ni\alpha\longmapsto\gamma_{\alpha,N}\in(\mathcal{C}([0,1],\mathbb{C}),\mathcal{B}_{\mathcal{C}})$, construct the induced measure

$$P_\beta^{(N)}(\mathcal{A}):=\lambda\left(\left\{\alpha\in[0,1]:\ \gamma_N^{\alpha,\beta}\in\mathcal{A}\right\}\right),\qquad \mathcal{A}\in\mathcal{B}_{\mathcal{C}}.$$

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- We have the canonical projection $\pi_{t_1,\dots,t_k}:\mathcal{C}([0,1],\mathbb{C})\to\mathbb{C}^k$,

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- **Theorem 1** can be rephrased as:

$$P_0^{(N)}\pi_{t_1,\dots,t_k}^{-1}\Longrightarrow P_{0;t_1,\dots,t_k}^{(k)}\qquad\text{as } N\to\infty.$$

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- (Joint work with Jens Marklof). For $\beta\notin\mathbb{Q}$, we prove the analogous limit theorem for $t\mapsto N^{-\frac{1}{2}}S_{t,N}(\alpha,\beta)$. The measure $\tilde{P}_\beta=\tilde{P}\neq\tilde{P}_0$ does not depend on β. We want to understand how a $\tilde{P}_0$ (or $\tilde{P}$)-typical curve "looks like".

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- An alternative approach uses equidistribution of long, closed horocycles in the unit tangent bundle of a suitably defined non-compact hyperbolic manifold. In particular, it provides an alternative probability space $(\mathcal{M}, \bar{\mu})$ (a homogeneous space) and realizes the curlicue process as an "explicit" function of the geodesic flow on $\mathcal{M}$.
  This approach extends from $\beta = 0$ to general β.

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- $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \simeq \mathfrak{H} \times \mathbb{R}$. Let $\Gamma_0 = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \right\rangle$.

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$\Gamma = \widetilde{\Gamma}_0 \ltimes \ldots$
- $\mathcal{M} = \Gamma\backslash G =$ a bundle over $\mathcal{M}_0 = \widetilde{\Gamma}_0\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$.
$\mathcal{M}_0$ is a 4-fold cover of T^1M, where $M = \Gamma_0\backslash\mathfrak{H}$ is a non-compact hyperbolic surface with three cusps $(0, \pm\frac{1}{2}, \infty)$ and finite area. Let us normalize $\overline{\mu} = \mu/\mu(\mathcal{M})$.

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- The geodesic flow Φ^s on $\mathcal{M}$ is given by right multiplication by $\left(\begin{pmatrix} e^{-s/2} & 0 \\ 0 & e^{s/2} \end{pmatrix}, 0; \mathbf{0}, 0\right) \in G$.

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$\Theta_f(z,\phi;\boldsymbol{\xi},\zeta) :=$
$y^{\frac{1}{4}} e(\zeta - \frac{1}{2}\xi_1\xi_2) \sum_{n\in\mathbb{Z}} f_\phi\left((n-\xi_2)y^{\frac{1}{2}}\right) e\left(\frac{1}{2}(n-\xi_2)^2 x + n\xi_1\right),$

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- where

$$f_\phi(w) := \begin{cases} e(-\sigma_\phi/8) f(w) & (\phi \equiv 0 \mod 2\pi) \\ e(-\sigma_\phi/8) f(-w) & (\phi \equiv \pi \mod 2\pi) \\ \frac{e(-\sigma_\phi/8)}{|\sin\phi|^{1/2}} \int_{\mathbb{R}} e\left(\frac{\frac{1}{2}(w^2 + w'^2)\cos\phi - ww'}{\sin\phi}\right) f(w')\mathrm{d}w' & \\ & (\phi \not\equiv 0 \mod \pi) \end{cases}$$

$$\sigma_\phi := \begin{cases} 2\nu, & \text{if } \phi = \nu\pi,\ \nu \in \mathbb{Z}; \\ 2\nu + 1, & \text{if } \nu\pi < \phi < (\nu+1)\pi,\ \nu \in \mathbb{Z}. \end{cases}$$

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- We are interested in Θ_χ, where $\chi = 1_{(0,1)}$.
- The limit curlicue process for $\beta \notin \mathbb{Q}$ can be described as

$$[0,1] \ni t \longmapsto \gamma_g(t) = \frac{\sqrt{t}}{\mu(\mathcal{M})}\Theta_\chi(g\Phi^s), \qquad s = 2logt, \qquad g \in \mathcal{M},$$

where $g \in \mathcal{M}$ is $\bar{\mu}$-random.
For $\beta = 0$ we have

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- In other words: the curlicue process shares many properties with the Brownian motion, even if it has dependent increments and its finite dimensional distribution are not Gaussian.

Thank You!