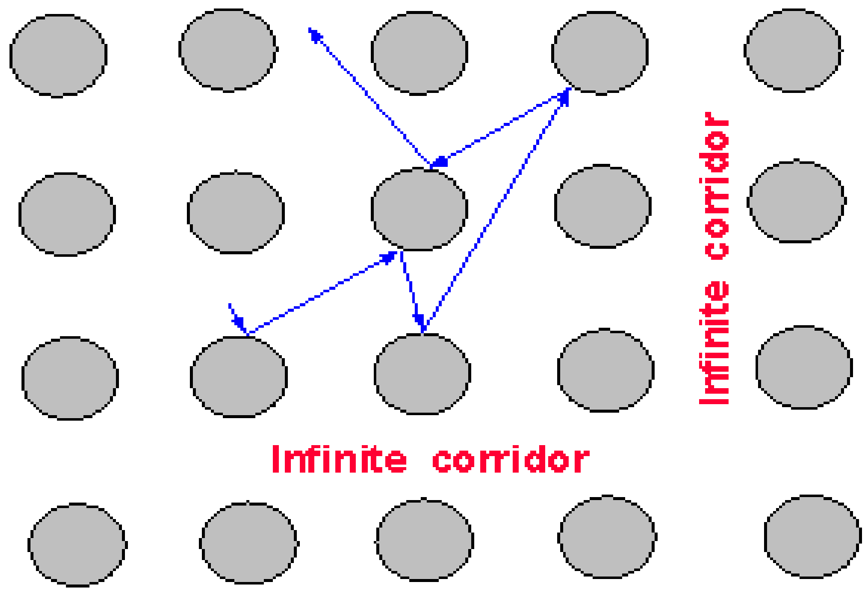
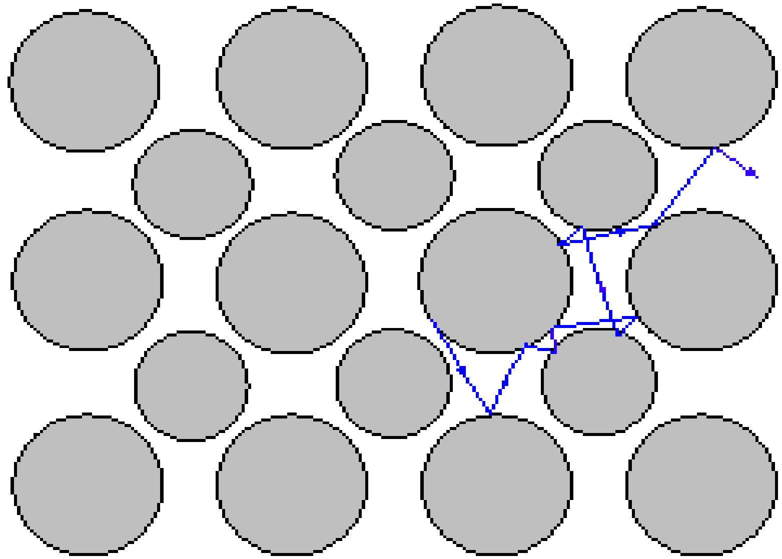
Periodic Lorentz gas



One moving particle bounces off a periodic array of fixed convex scatterers.

Shown above: the “infinite horizon” case

“Finite horizon” case:



Larger scatterers blocking the particle from all sides.

No infinite corridors.

Symbols and notation

$q(t)$ position of the particle at time t

$v(t)$ velocity of the particle at time t

q_n position of the particle at nth collision

v_n velocity of the particle at nth collision

No external forces, **finite** horizon:

$q(t) \sim N(0, D_0 t)$ as $t \to \infty$ (asymptotically normal)

D_0 is diffusion matrix (integral of autocorrelations)

$$D_0 = \int_{\mathbf{R}} \langle v(t)\, v(0) \rangle_\mu \, dt$$

$q_n \sim N(0, \hat{D}_0 n)$ as $n \to \infty$ (asymptotically normal)

$\hat{D}_0$ is diffusion matrix (infinite sum of autocorrelations)

$$\hat{D}_0 = \Sigma \langle dq_n \, dq_0 \rangle_\nu$$

$\hat{D}_0 = \tau D_0$ where τ is the mean free path

Sinai & Bunimovich 1981

No external forces, **infinite** horizon:

$q(t) \sim N(0, D_1 t \log t)$ as $t \to \infty$ (superdiffusive)

D_1 is superdiffusion matrix

Chernov & Dolgopyat 2009

$q_n \sim N(0, \hat{D}_1 n \log n)$ as $n \to \infty$ (superdiffusive)

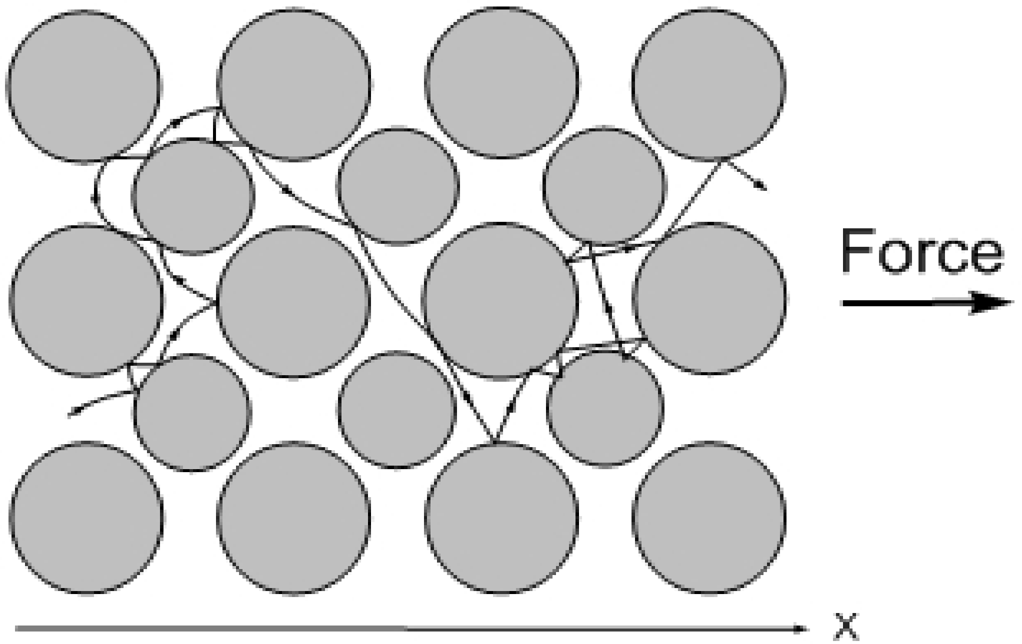
$\hat{D}_1$ is superdiffusion matrix

(finite sum of parameters of corridors)

Szász & Varjú 2007

$\hat{D}_1 = \tau D_1$ where τ is the mean free path

Lorentz gas with a constant external force



Particle ("electron") is subject to an external ("electric") field $E = (\varepsilon, 0)$ directed horizontally

$\varepsilon > 0$ is small

Lorentz gas with external field:

Equations of motion

$$dq/dt = v \qquad dv/dt = E$$

Total energy (kinetic + potential) is preserved:

$$\tfrac{1}{2} v^2 - \varepsilon x = \text{const}$$

Thus when the particle is driven by the field and $x(t)$ grows, then $v(t)$ has to grow, too: $v^2 = O(x)$

This is unrealistic for an actual electrical current.

Electrons are expected to travel at a linear rate, i.e. $\langle x \rangle = Jt$,

where J represents the current

Lorentz gas with a thermostat

Electrons move subject to a **force** and **thermostat**:

$$dq/dt = v \qquad dv/dt = E - \langle E,v\rangle v$$

Gaussian thermostat

(Moran & Hoover 1987)

Lorentz gas with a thermostat

Electrons move subject to a force and thermostat:

$$dq/dt = v \qquad dv/dt = E - \langle E,v \rangle v$$

Now $\langle v,v \rangle = 1$ at all times, because $\langle v, dv/dt \rangle = 0$

In other words, the kinetic energy is kept constant.

The extra term prevents the electrons from speeding (*heating up*) or slowing down (*cooling down*).

It keeps the *temperature* fixed. Hence its name:

thermostat.

Gaussian thermostat, **finite** horizon

Then $q(t) \sim J_0 t + N(0, D_0(\varepsilon)t)$

(drift + diffusion)

The electrical current J_0 satisfies

$$J_0 = \sigma_0 E + o(\varepsilon) \qquad \text{(Ohm's law)} \qquad \text{recall: } \varepsilon = |E|$$

Electrical conductivity σ_0 satisfies

$$\sigma_0 = \tfrac{1}{2} D_0 \qquad \text{(Einstein relation)}$$

D_0 is again the diffusion matrix (for the field-free process)

$$D_0(\varepsilon) = D_0 + o(1) \quad \text{as } \varepsilon \to 0$$

Note: the current J_0 is not always parallel to the field E

(this is known as Hall effect in physics)

Chernov, Eyink, Lebowitz, Sinai 1993 and Chernov, Dolgopyat 2009

Gaussian thermostat, **infinite** horizon

Assume: the field E is not parallel to any infinite corridor

Then $q(t) \sim J_1 t + N(0, D_1(\varepsilon)t)$

(drift + diffusion)

The electrical current J_1 does not satisfy Ohm's law:

$J_1 = \sigma_1 E\,|\log \varepsilon| + O(\varepsilon)$ recall: $\varepsilon = |E|$

Superconductivity coefficient σ_1 satisfies

$\sigma_1 = ½D_1$ (Einstein relation still holds)

D_1 is again the diffusion matrix (for the field-free process)

$D_1(\varepsilon) = D_1\,|\log \varepsilon| + O(1)$

Chernov, Dolgopyat 2009

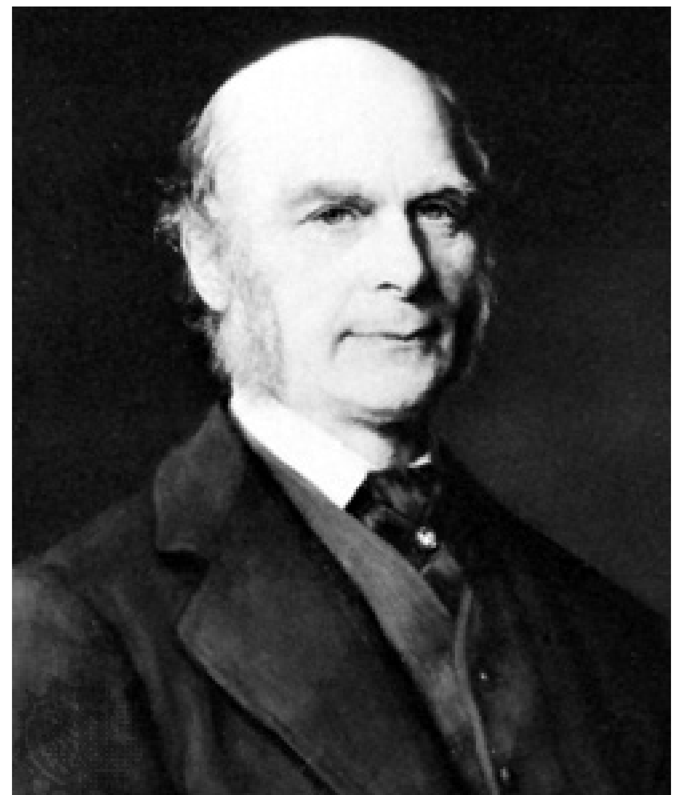
Back to Lorentz gas with external field

No thermostat is imposed anymore. Questions:

Describe asymptotic behavior of the position and velocity of the particle as time $t \to \infty$.

Equivalent to **Galton board**

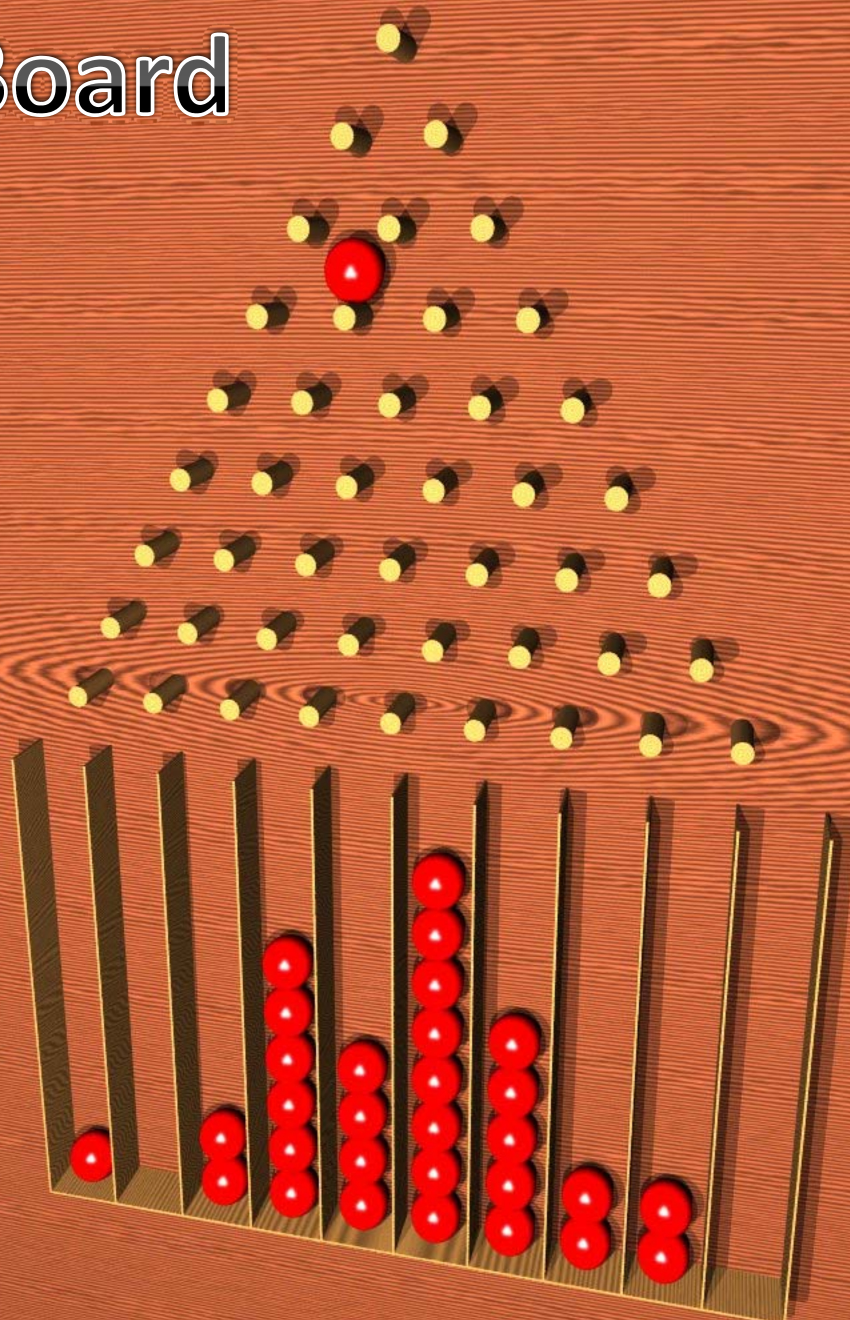
An upright board with a periodic array of fixed pegs on which balls are rolling down bouncing off the pegs



Introduced by Sir Francis Galton (1822-1911)

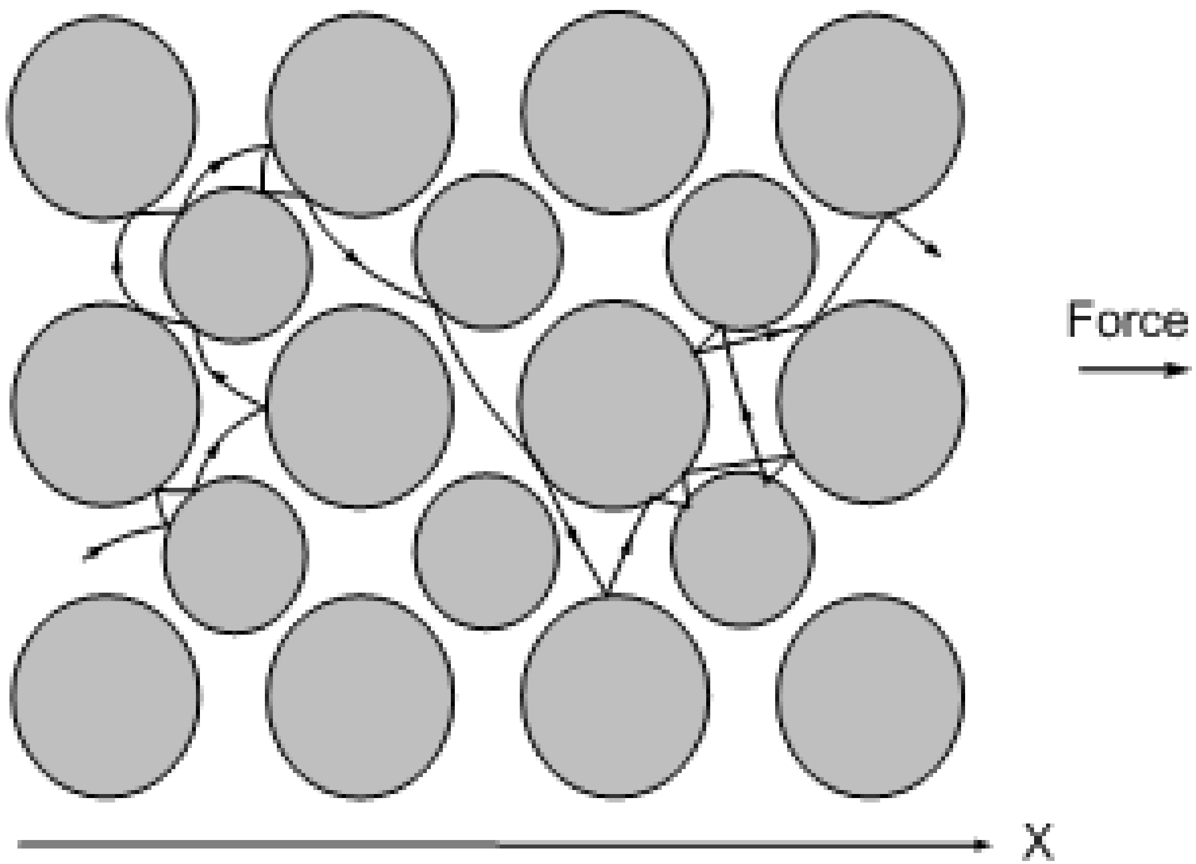
Resembles a modern pinball machine

Galton Board



Difficulties:

- Particle accelerates as it moves away
- Phase space is not compact, invariant measure is infinite
- Initial distribution is concentrated in a compact domain (say, $0 \le x, y \le 1$)
- Images of the initial measure escape to infinity
- Dynamics is inhomogeneous in time and space (speed increases, trajectory straightens)



Force
X

For the Lorentz gas/Galton board with a constant external field and **finite horizon**

Conjectures in physics literature 1979-2008 (based on heuristic and empirical studies):

Position $x(t) \sim t^{2/3}$ Velocity $v(t) \sim t^{1/3}$

Note: the electron travels at a slow (<u>sublinear)</u> rate.

The reason is: backscattering ("Fermi acceleration")

Chernov & Dolgopyat 2009:

Average position $x(t)$ does grow as $t^{2/3}$

Average velocity $v(t)$ does grow as $t^{1/3}$

Rescaled position $t^{-2/3}x(t)$ has a limit distribution

Rescaled velocity $t^{-1/3}v(t)$ has a limit distribution

Rescaled position converges to an Itô diffusion process satisfying certain Stochastic Differential Equations

Chernov & Dolgopyat 2009:

The limit stochastic process is recurrent (comes back to x=0 infinitely many times).

The original trajectory $x(t)$ is recurrent, too: the particle's coordinate returns to its initial value $x(0)$ infinitely many times with probability one.

A surprising fact, but intuitively follows from the invariance of an infinite measure

Lorentz gas with external field and **infinite horizon**

This remains an open problem.

We (C&D) are currently working on it.

Our conjectures:

Position $x(t) \sim (t \log t)^{2/3}$

Velocity $v(t) \sim (t \log t)^{1/3}$

The finite horizon Galton board was studied via approximating it by the Lorentz gas with Gaussian thermostat.

Both are ε–perturbations of the field-free (billiard) dynamics, and they are ε^2–close to each other.

So knowing one, we can effectively study the other.

For the infinite horizon Galton board this approach fails. Here is the reason:

The trajectories with and without Gaussian thermostat are actually $(\varepsilon^2 t^3)$–close to each other, where t is the time between collisions.

In finite horizon, $t=O(1)$, so we have ε^2–closeness

In infinite horizon, $t=O(\varepsilon^{-1/2})$, so we only have $\varepsilon^{1/2}$–closeness, which is very poor.

So we introduce a **new thermostatted model**:

The particle moves under the constant field (along a parabola, with its speed growing) between collisions, but its energy is reset at each collision. We call this **thermostatted walls**. By the way, this is a more physically sensible thermostat

(Gaussian thermostat was criticized by many as unrealistic).

But it causes unforeseen and peculiar complications: **the dynamics ceases to be invertible**.

- Some phases points may have <u>more than one</u> preimage (*indeterminate past*).
- Some phase points may have <u>no</u> preimages at all (*no past*).

To visualize the situation:

Let F: $\mathbb{T}\to\mathbb{T}$ be a hyperbolic automorphism of a 2-torus.

Let $\mathbb{T}=M_1$ Ú···Ú M_k be a partition of $\mathbb{T}$ into domains with piecewise smooth boundaries.

Let G: $\mathbb{T}\to\mathbb{T}$ be a map that is smooth on each M_i and its restriction to M_i is a C^2-perturbation of the identity map on M_i.

Then the composition $F \circ G$ is a map that has strong expansion and contraction, but the images of M_i may overlap and/or may leave uncovered gaps in $\mathbb{T}$.

Such maps were studied recently by operator technique

Baladi & Gouëzel 2009 and 2010

We (C&D) use standard pairs, Growth Lemma, and Coupling Lemma to:

Prove the existence and uniqueness of a physically observable (SRB-like) measure.

Establish exponential decay of correlations and limit theorems.

We only work with general unstable, i.e., expanding curves (we do not need unstable manifolds) and only iterate them forward.

Final results for the Lorentz gas with thermostatted walls:

All the limit theorems about the drift, (super)diffusion, (super)conductivity, etc., previously proven for the Gaussian thermostat are now proven for the thermostatted walls.

Both in finite and infinite horizon.

Chernov & Dolgopyat 2010