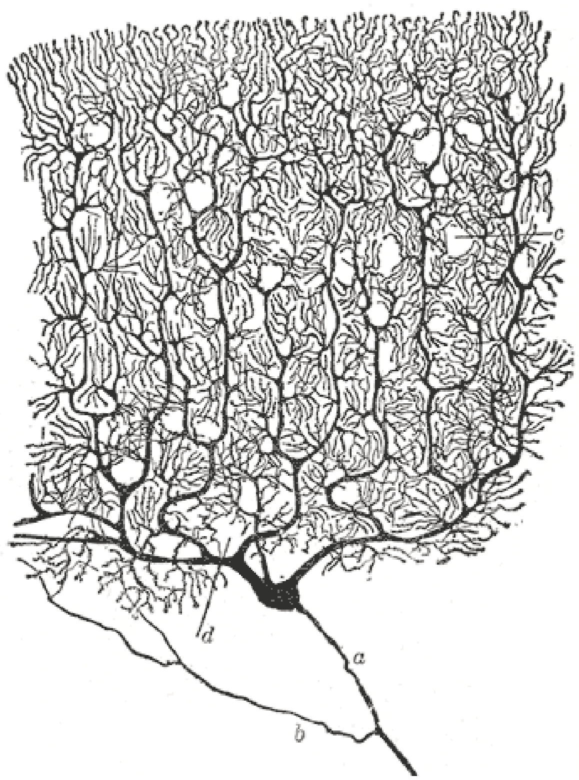
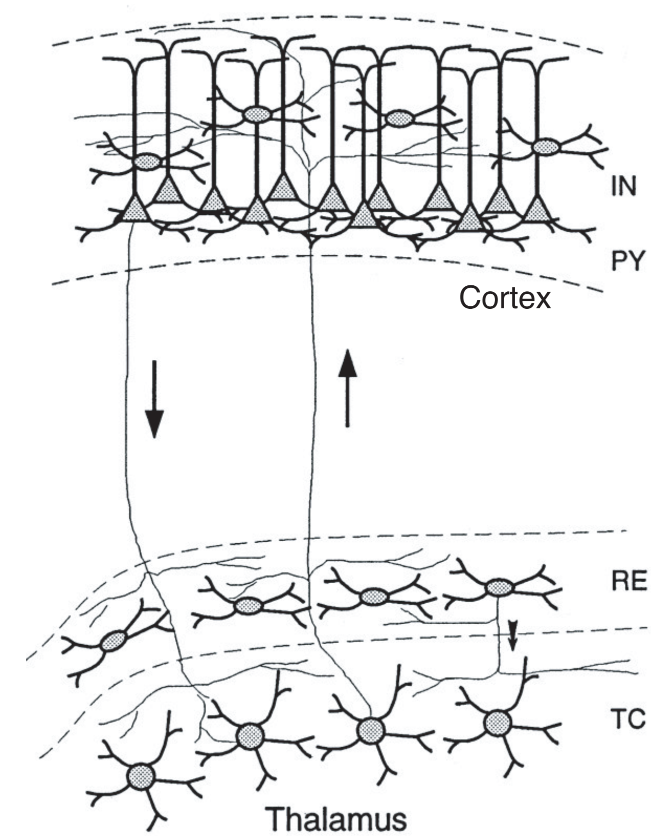
Mathematical Neuroscience: from neurons to networks



Part I

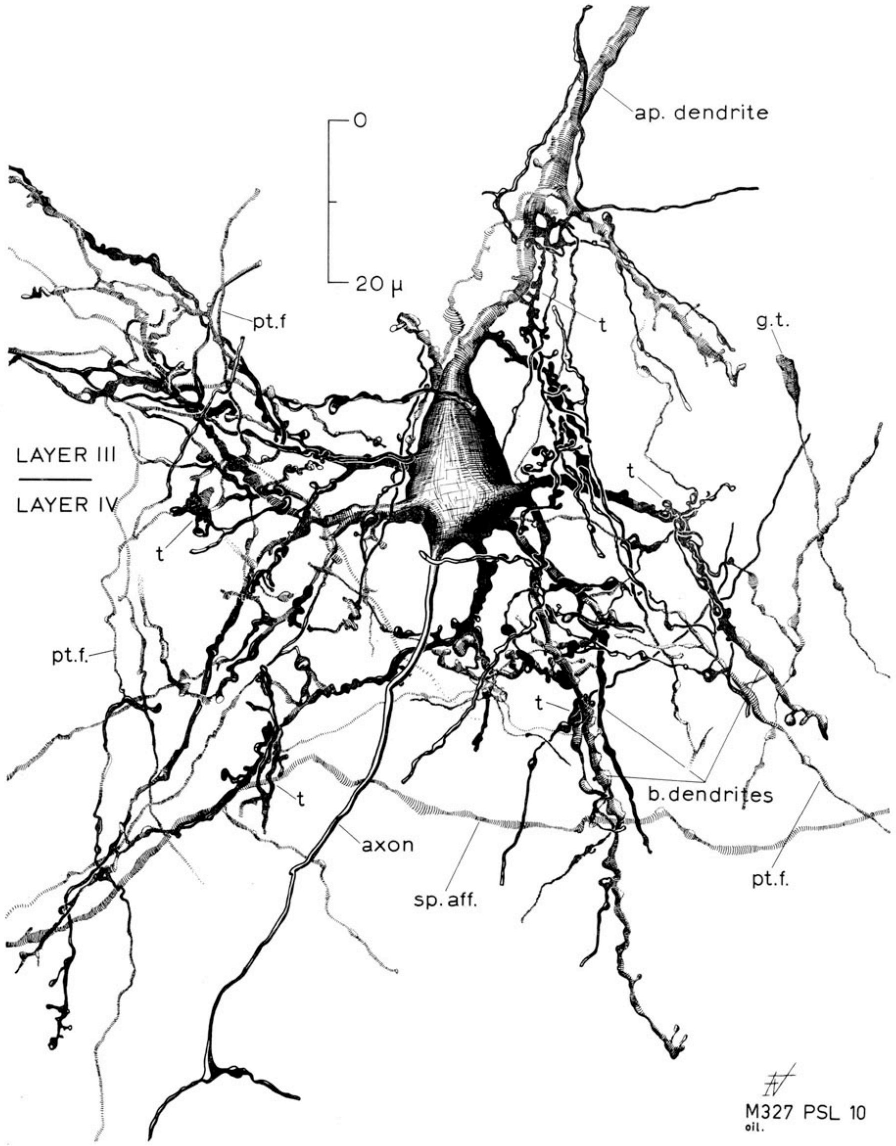
Steve Coombes

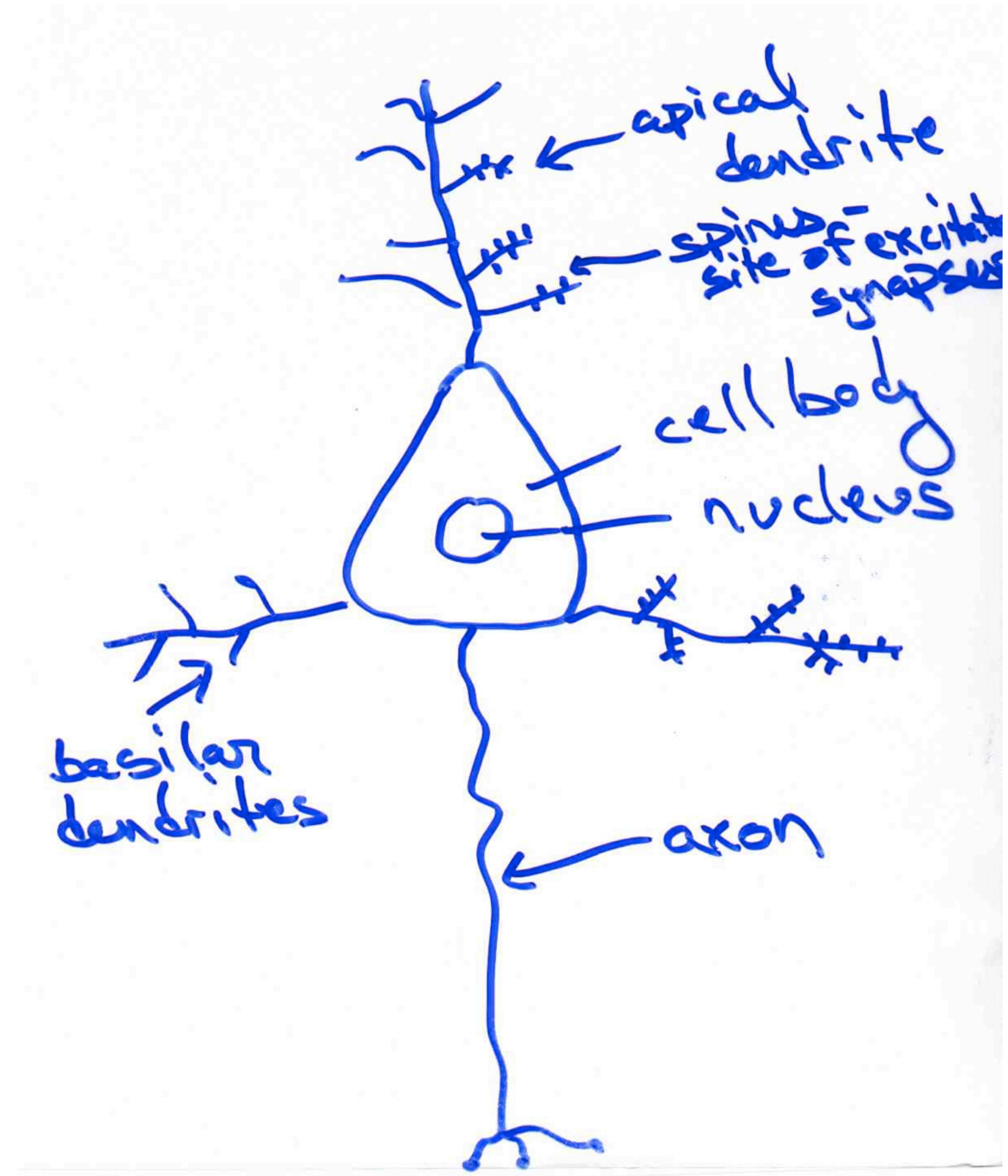


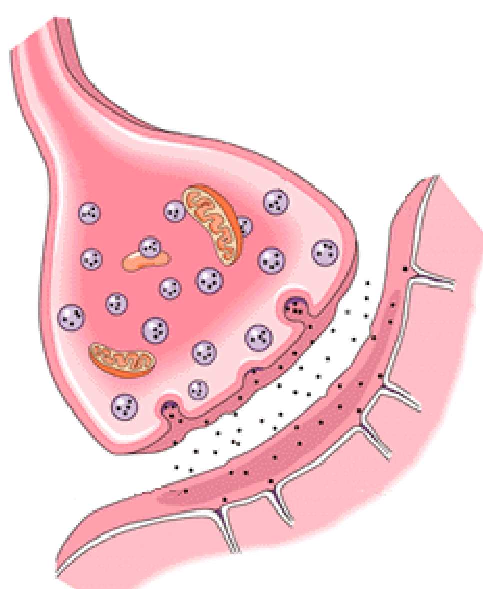
The University of Nottingham

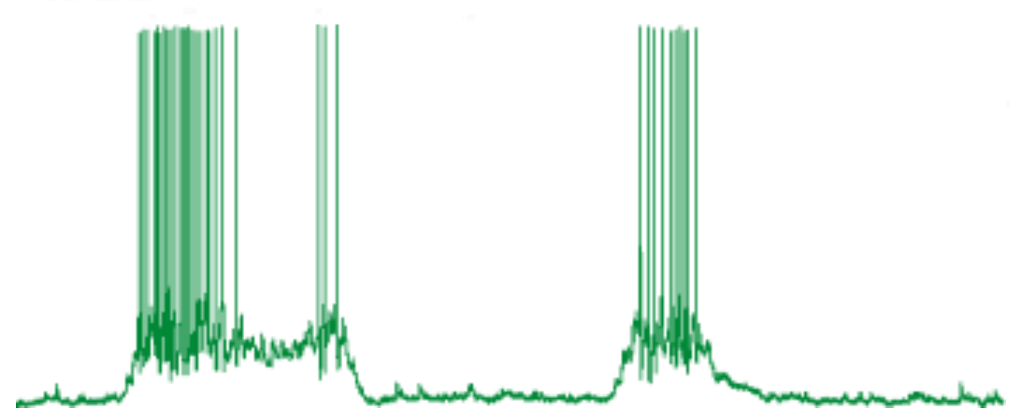
School of Mathematical Sciences

Neurons: pyramidal cells



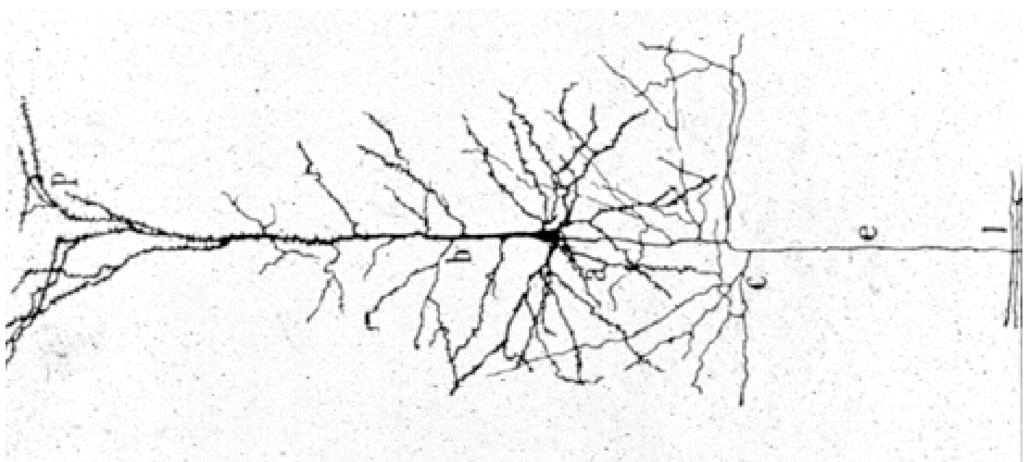


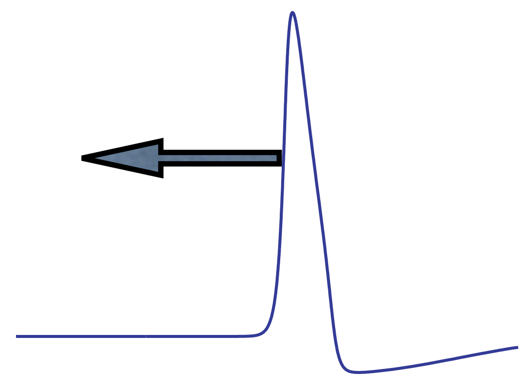




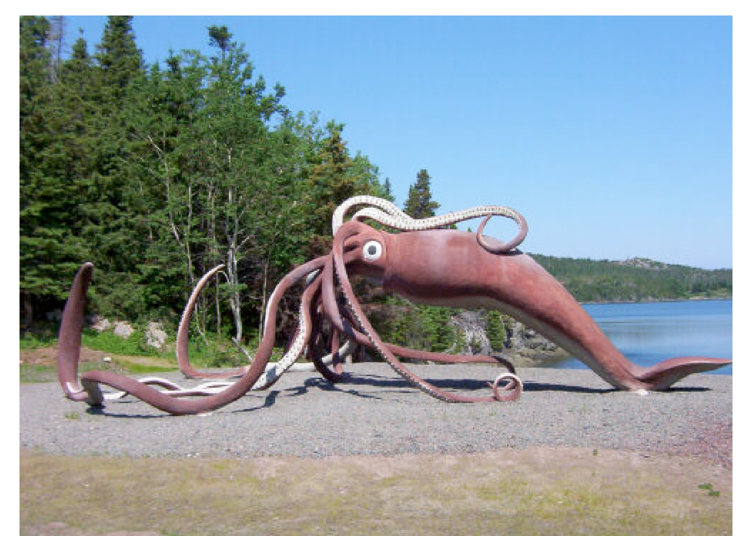
Hodgkin and Huxley

(1950s) express (and subsequently fit) the dynamics of gating variables (representing membrane channels) using the mathematical language of nonlinear ODEs.



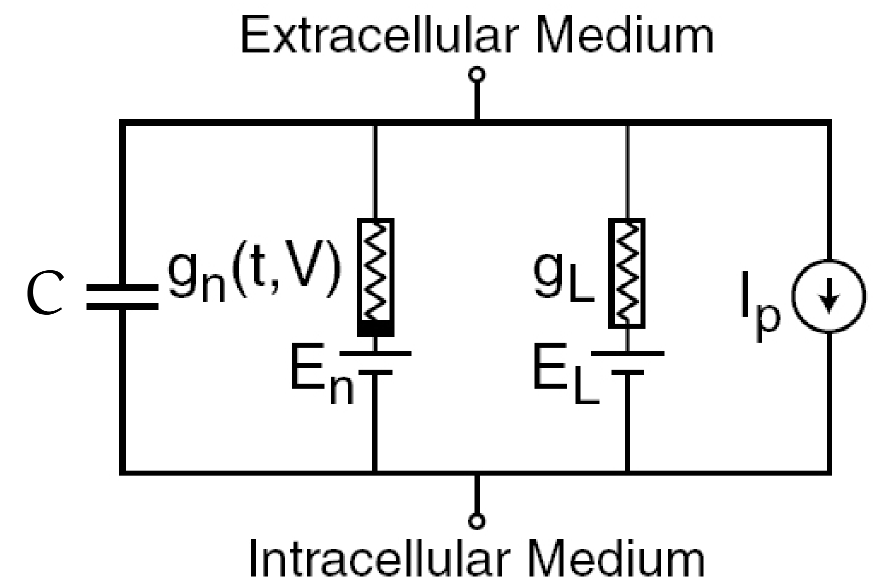


Action
potentials
m/s



Active membrane models

$$C\frac{dv}{dt} = -\sum_k g_k m_k^{p_k} h_k^{q_k}(v - v_k) + I$$



$p_k, q_k \in \mathbb{Z}$

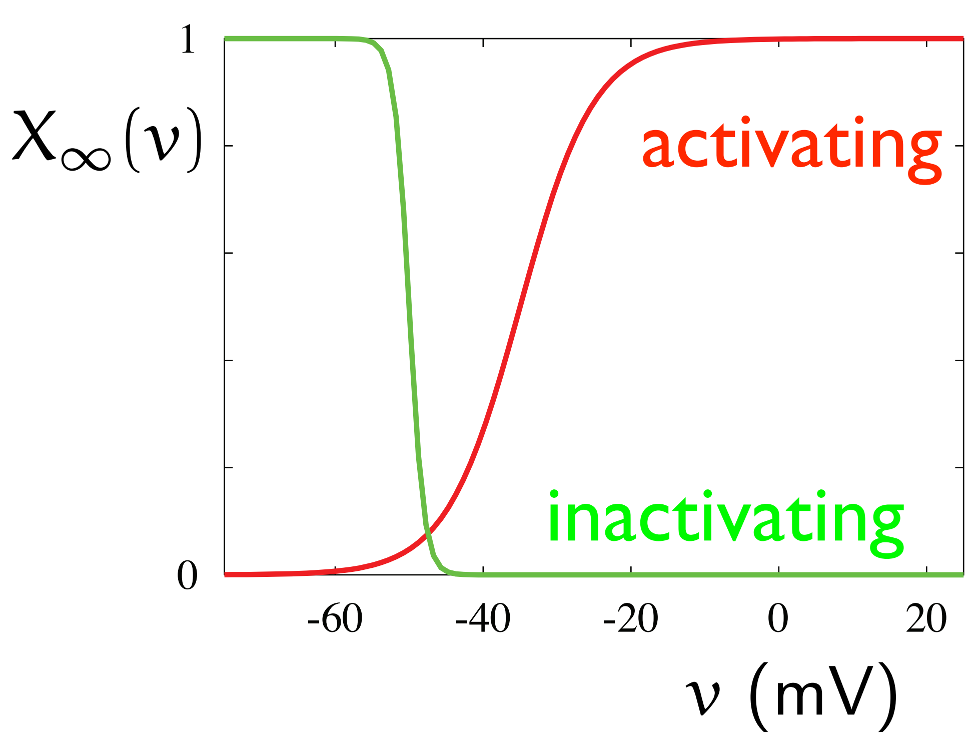
v – membrane potential

v_k – reversal potentials

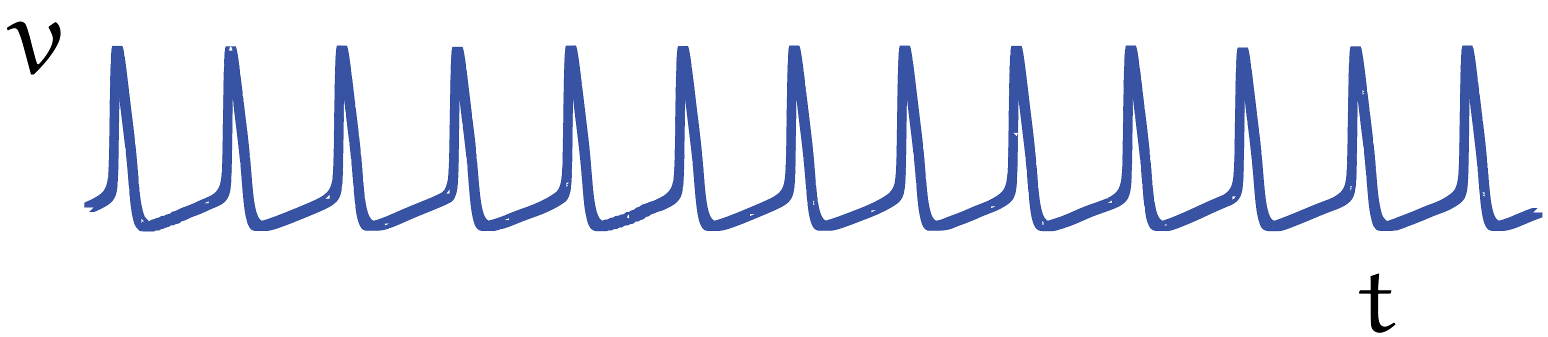
m_k, h_k – gating variables

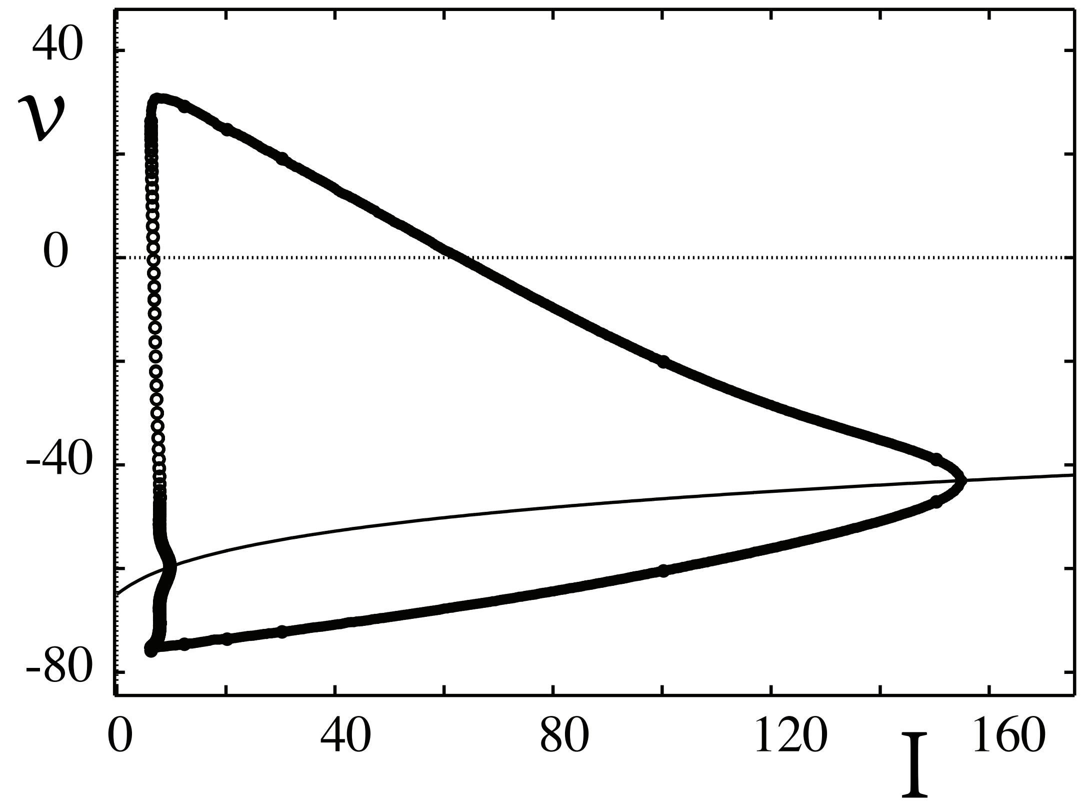
g_k – conductances

$$\frac{dX}{dt} = \frac{X_\infty(v) - X}{\tau_x(v)}$$



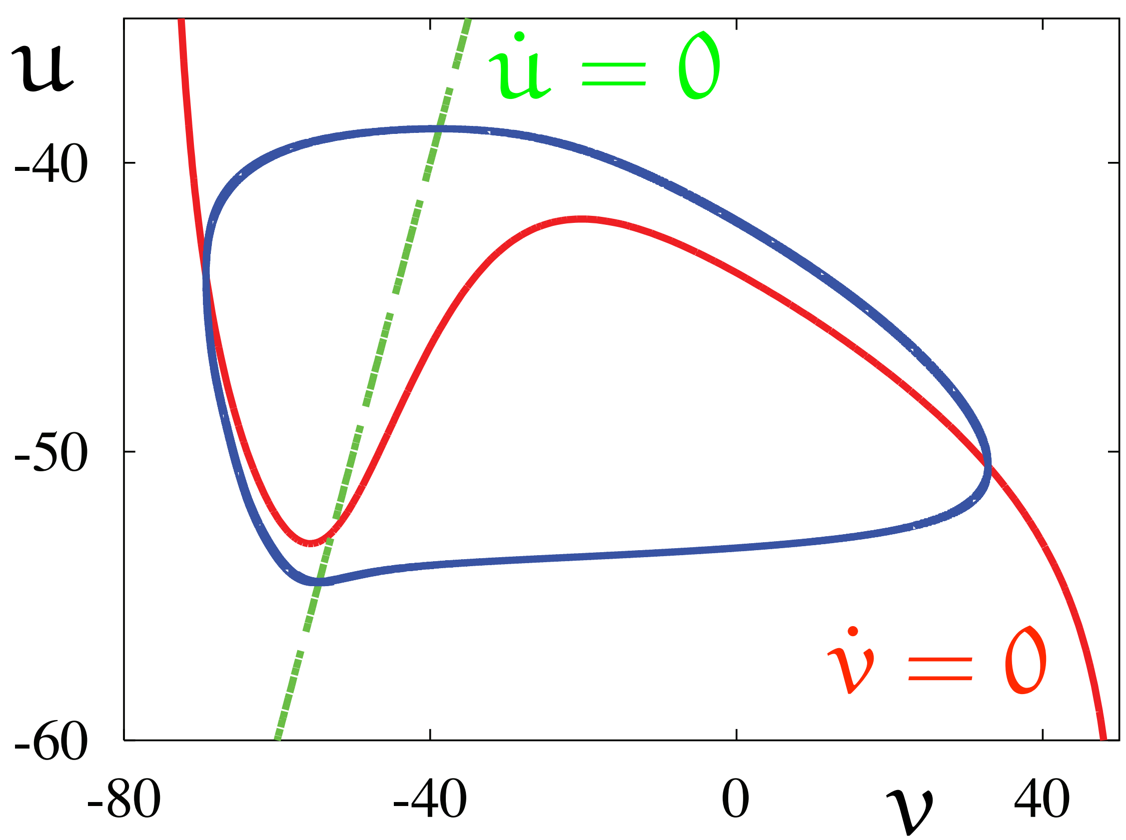
Hodgkin and Huxley: (v, m, n, h)





Reduction: $m \to m_\infty(v)$

$$(n, h) \to (n_\infty(u), h_\infty(u))$$

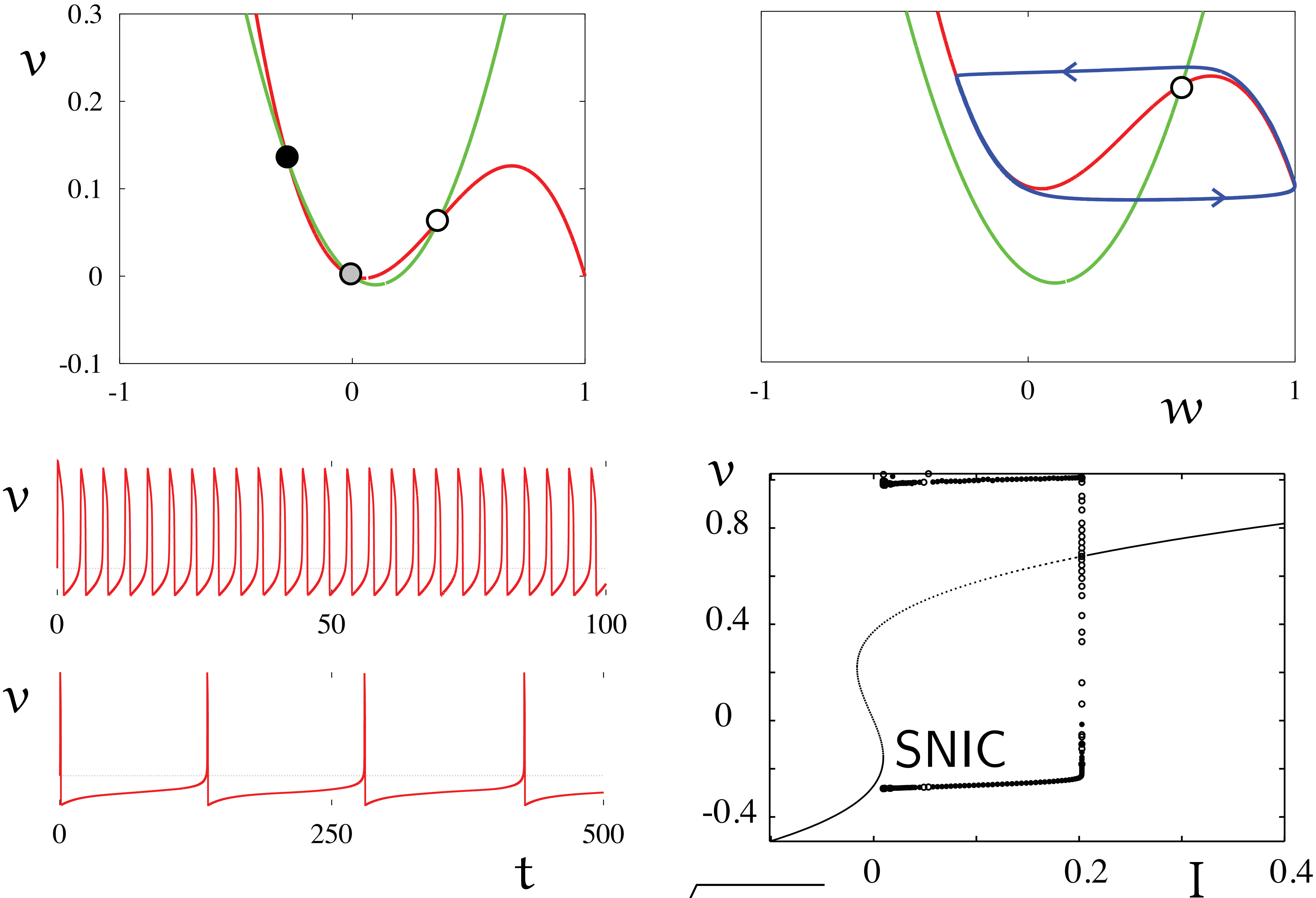


$$C\frac{dv}{dt} = f(v, u) + I$$

$$\frac{du}{dt} = g(v, u)$$

Method of equivalent potentials gives f and g in terms of HH model - Abbott and Kepler 1990

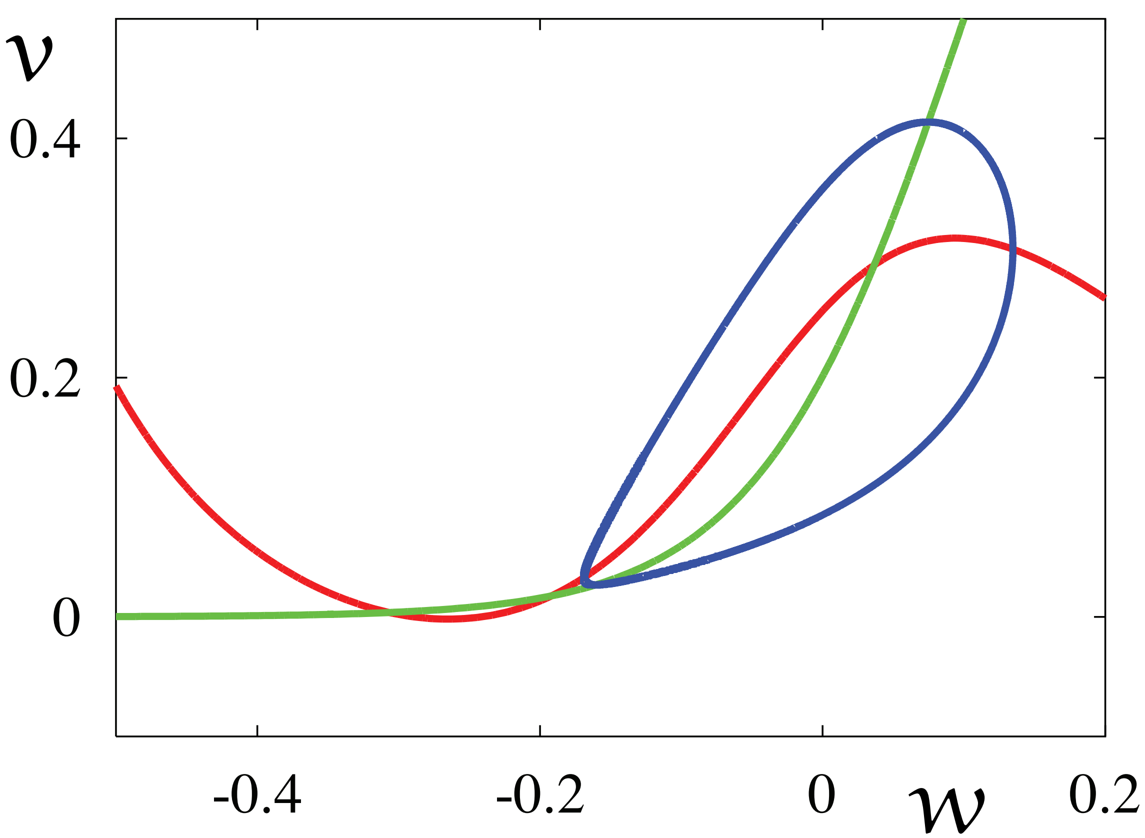
Cortical model (slow firing)

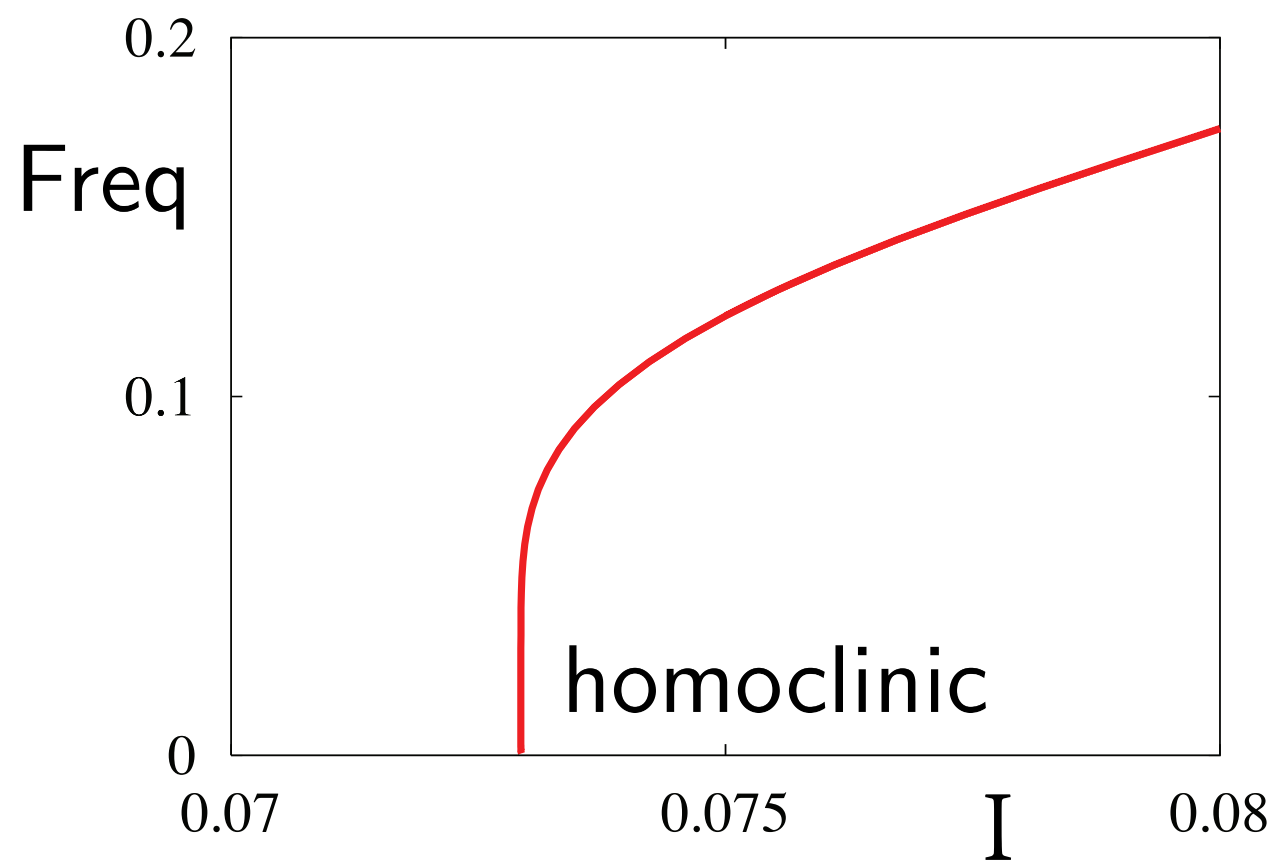


$$\text{Freq} \sim \sqrt{I - I_c}$$

Morris-Lecar model (slow firing) (v, w)

Originally a model of the barnacle giant muscle fiber

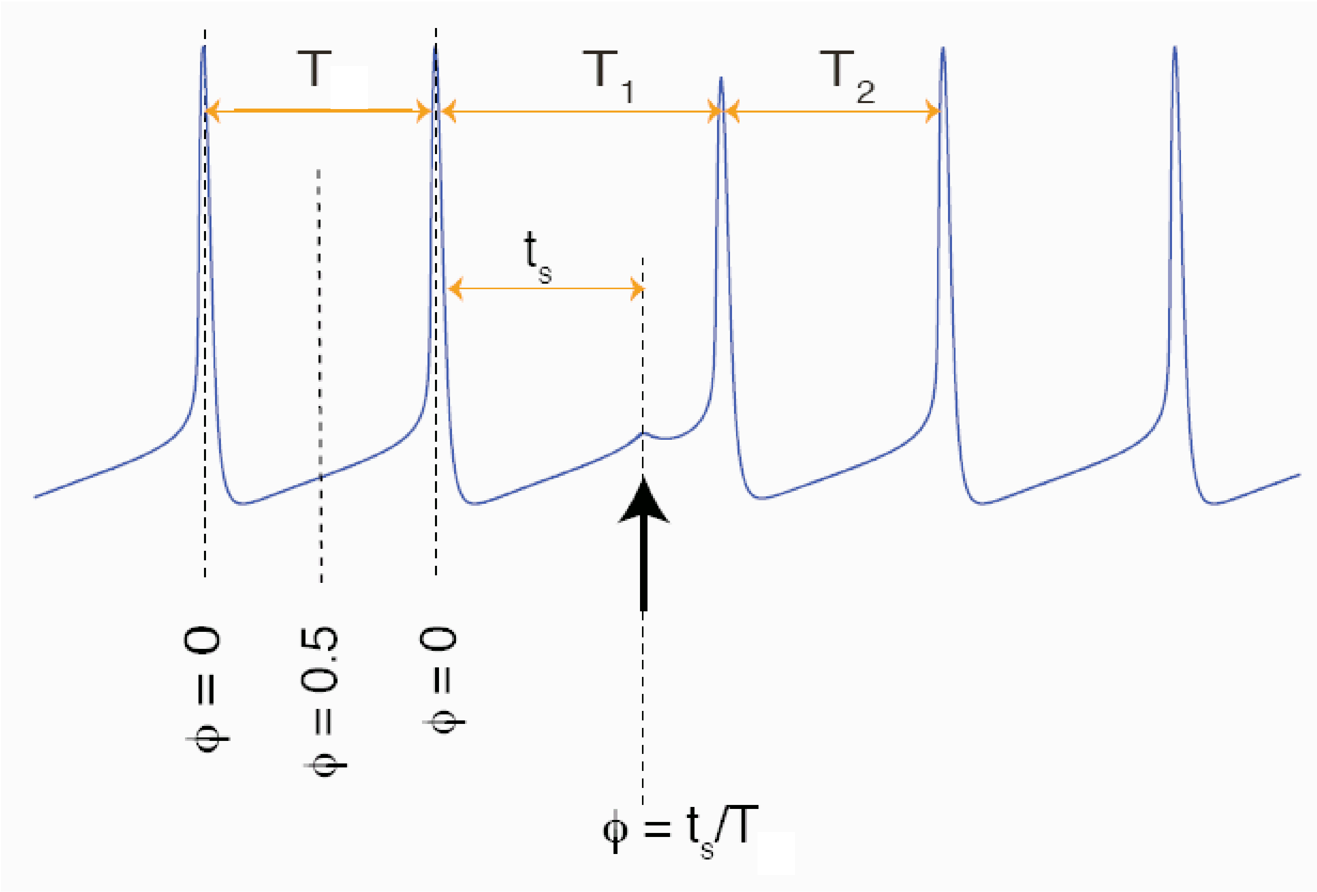


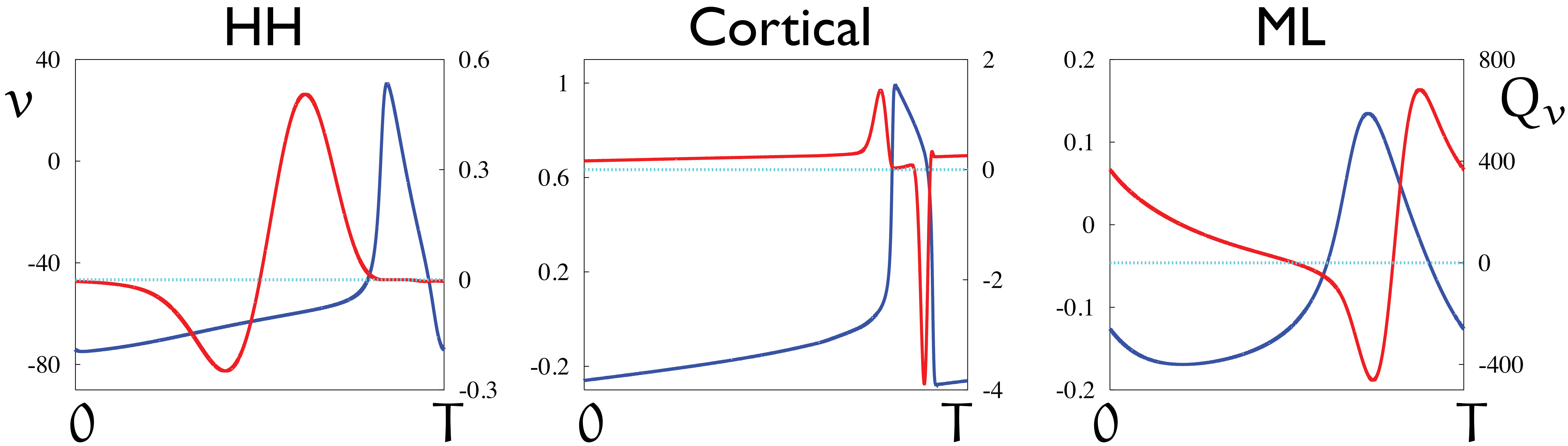


$$\text{Freq} \sim -\frac{1}{\ln(\mathrm{I} - \mathrm{I_c})}$$

Phase Response Curve (PRC)

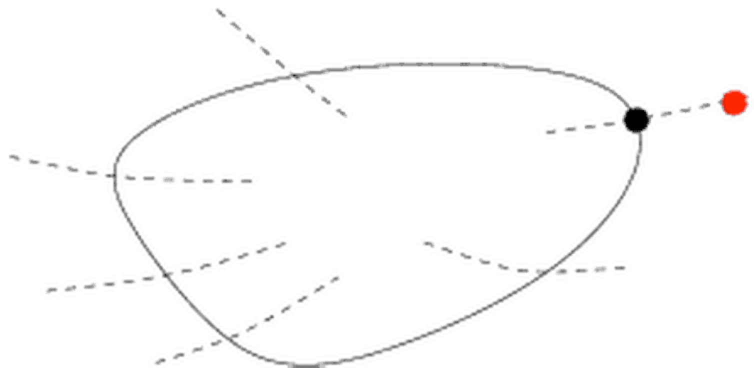
A **PRC** tabulates the transient change in the cycle period of an oscillator induced by a perturbation as a function of the phase at which it is received.



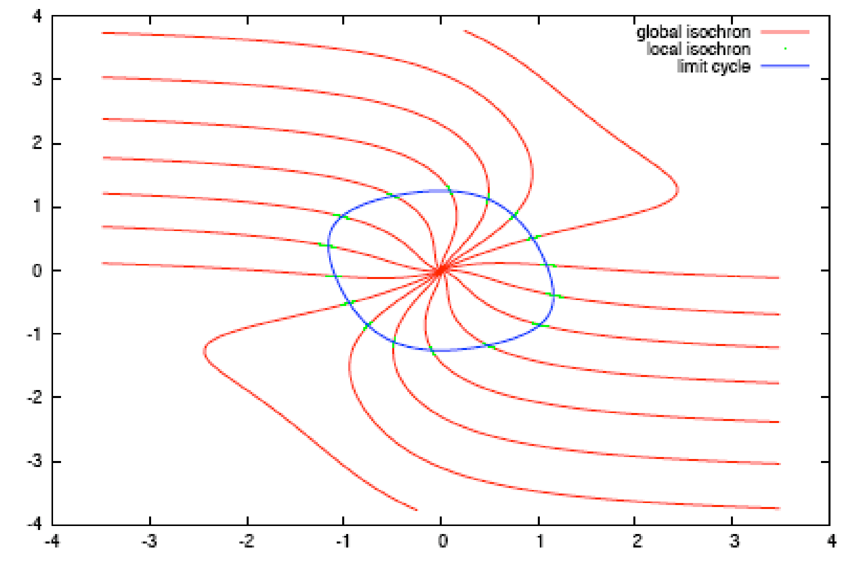


obtained numerically

$$Q = \nabla_Z \theta$$



Isochrons as leaves of the stable manifold of a hyperbolic limit cycle

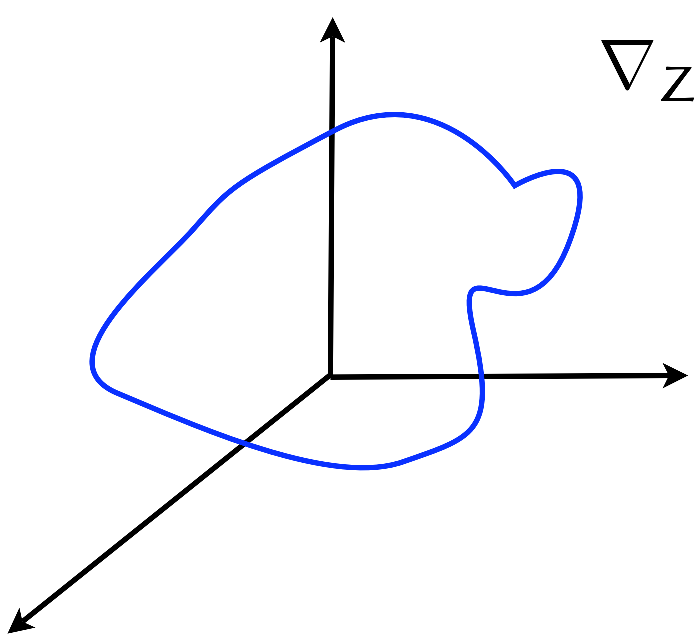


Call the orbit $z = Z(t)$ where $\dot{z} = F(z)$

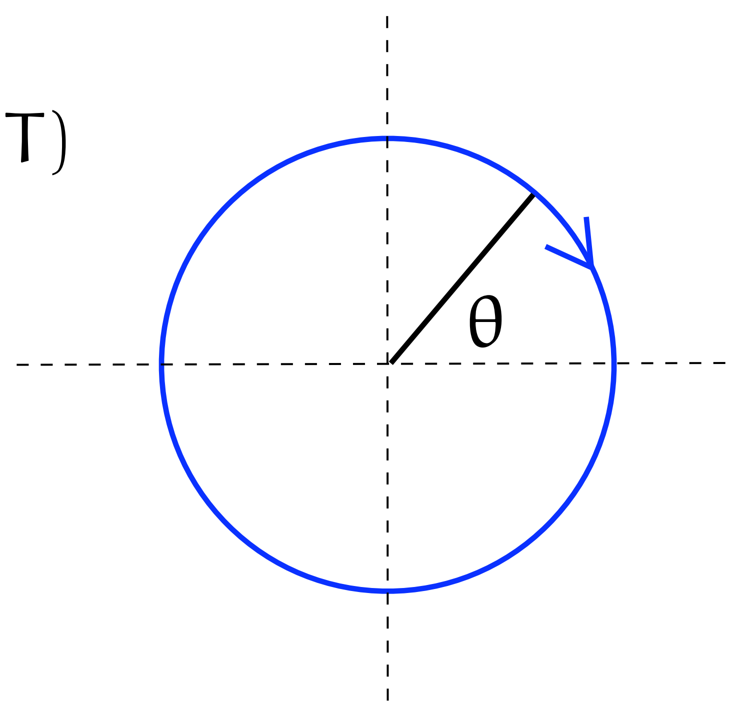
Introduce a phase (isochronal coordinates) θ

$$\frac{dQ}{dt} = D(t)Q, \qquad D(t) = -DF^T(Z(t))$$

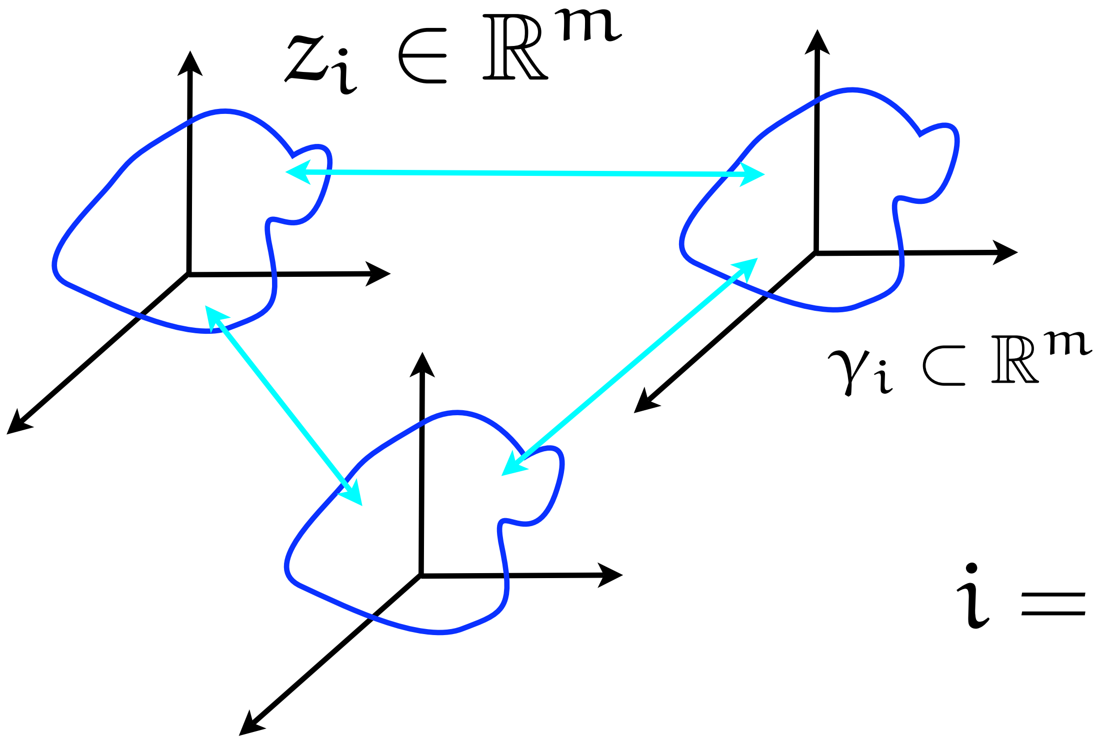
$$\nabla_{Z(0)} \cdot F(Z(0)) = \frac{1}{T} \text{ and } Q(t) = Q(t+T)$$



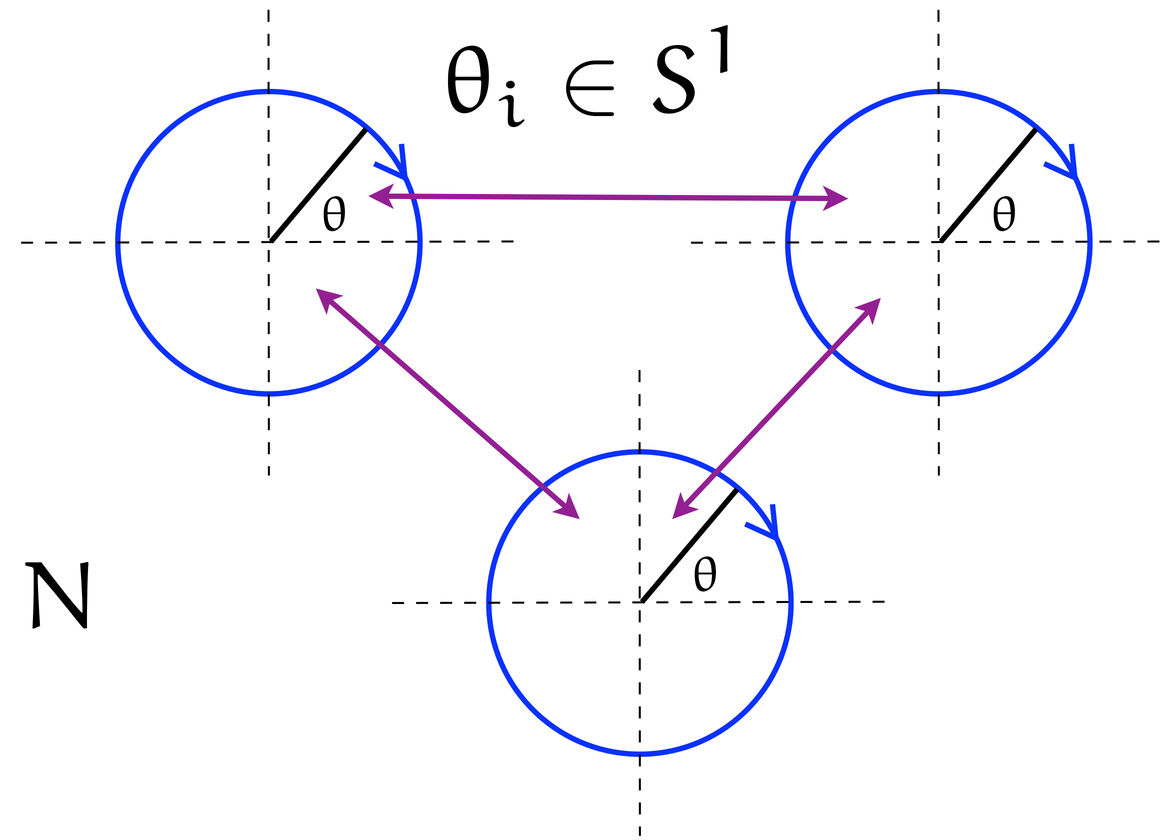
$$\dot{\theta} = \frac{1}{T}$$



Weak Coupling



$$i = 1, \ldots, N$$



$$\dot{z}_i = F(z_i) + \epsilon G_i(z_1, \ldots, z_N)$$

Uncoupled system has an exponentially stable limit cycle γ_i

Direct product of hyperbolic limit cycles is a normally hyperbolic invariant manifold

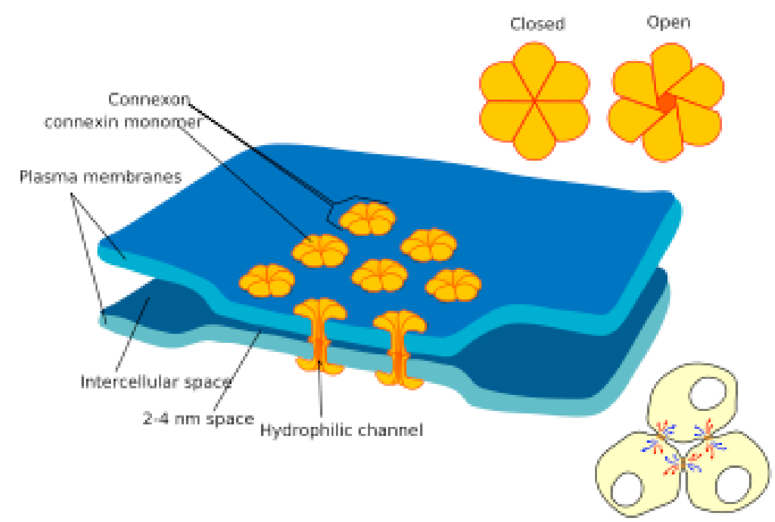
$$\dot{\theta}_i = \frac{1}{T} + \epsilon \left\langle Q(\theta_i), G_i(\Gamma(\theta)) \right\rangle$$

Drive

PRC

Coupled oscillator networks

An example: gap junction coupling



$$\frac{1}{N}\sum_{j=1}^{N}(v_j - v_i)$$

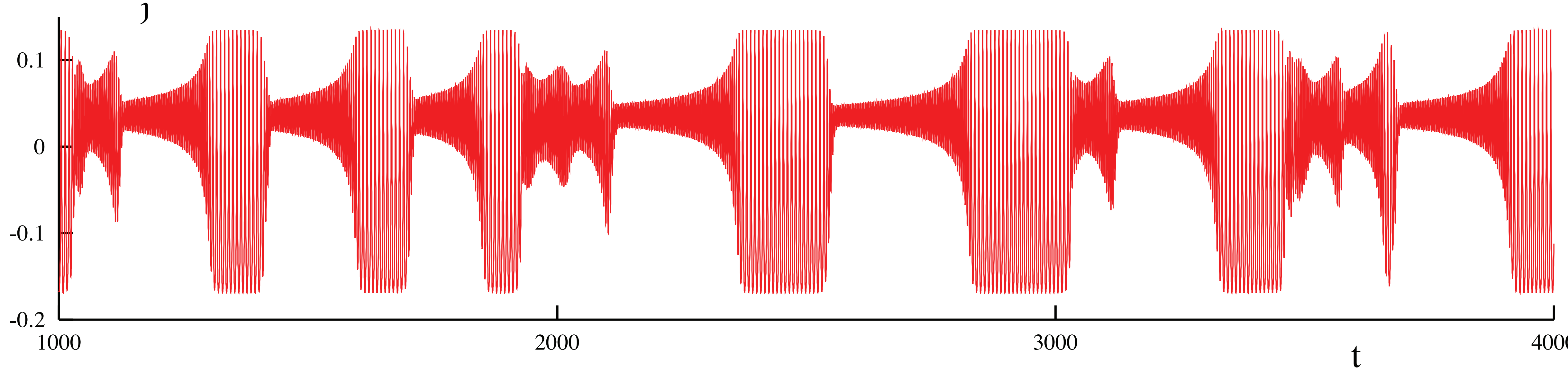
Averaging gives $$H(\theta) = \frac{1}{T}\int_0^T \langle Q(t), (v(t+\theta T) - v(t), 0)\rangle \, dt$$

Kopell and Ermentrout

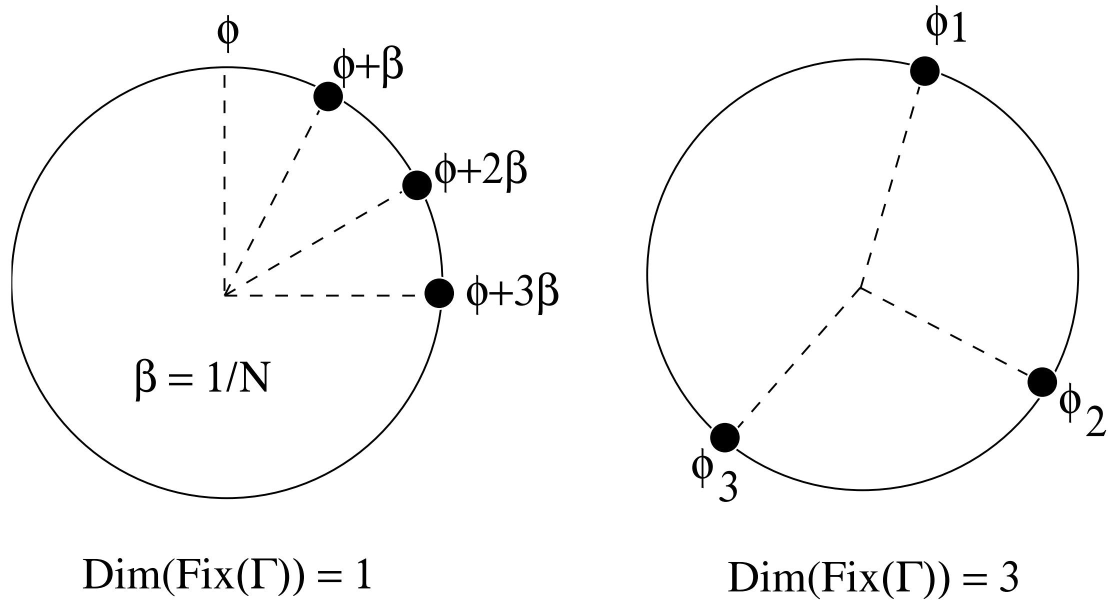
$$\dot{\theta}_i = \frac{1}{T} + \frac{\epsilon}{N}\sum_j H(\theta_j - \theta_i)$$

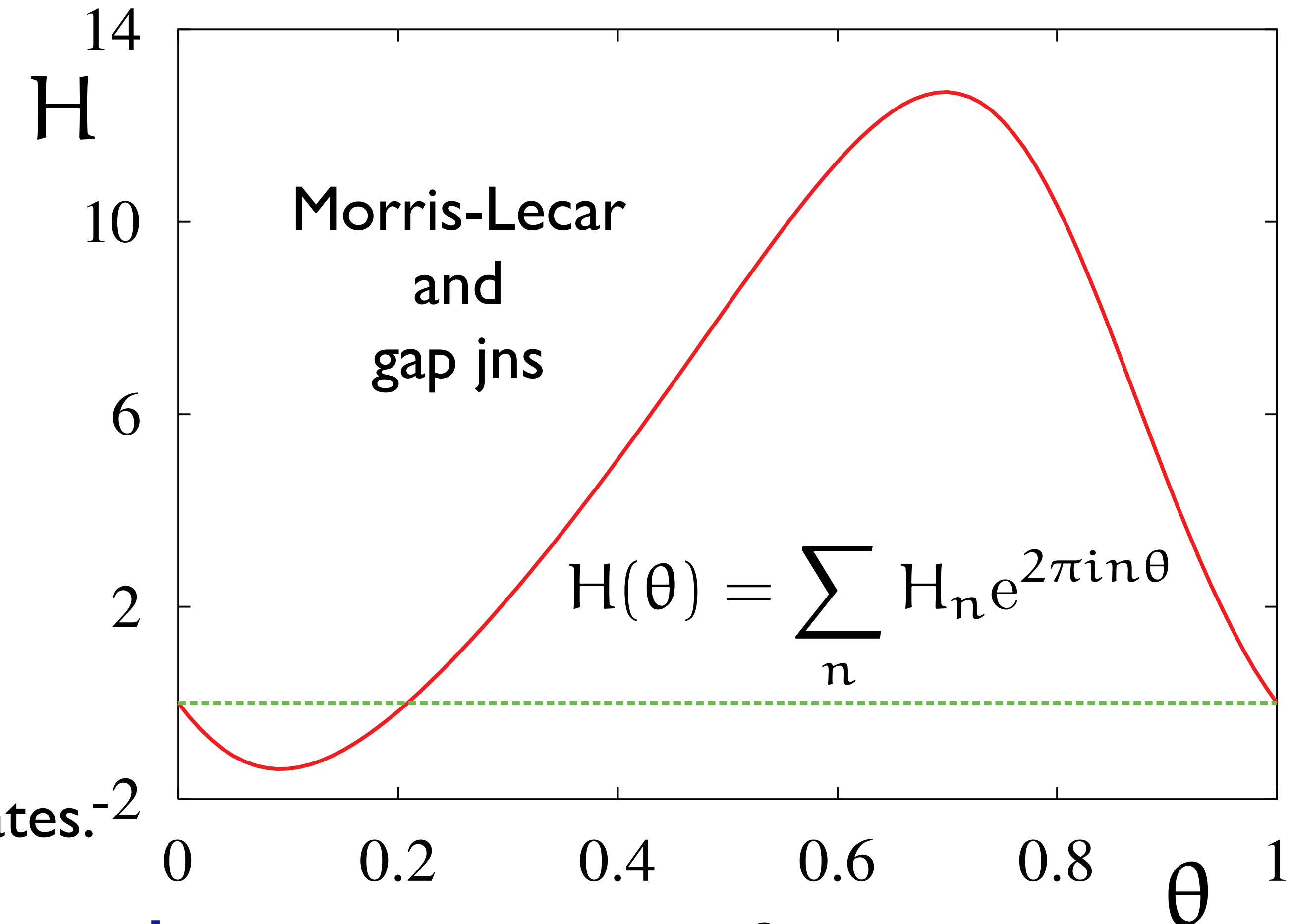
$$E = \frac{1}{N}\sum_j v_j$$

Morris-Lecar



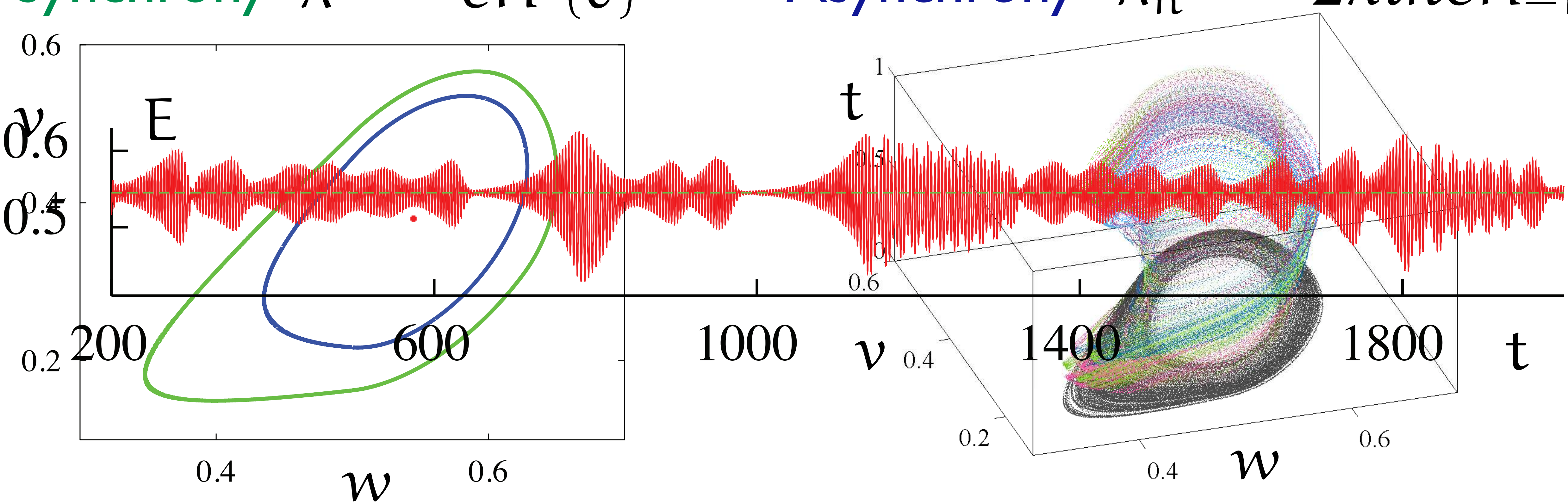
Stability of phase-locked states





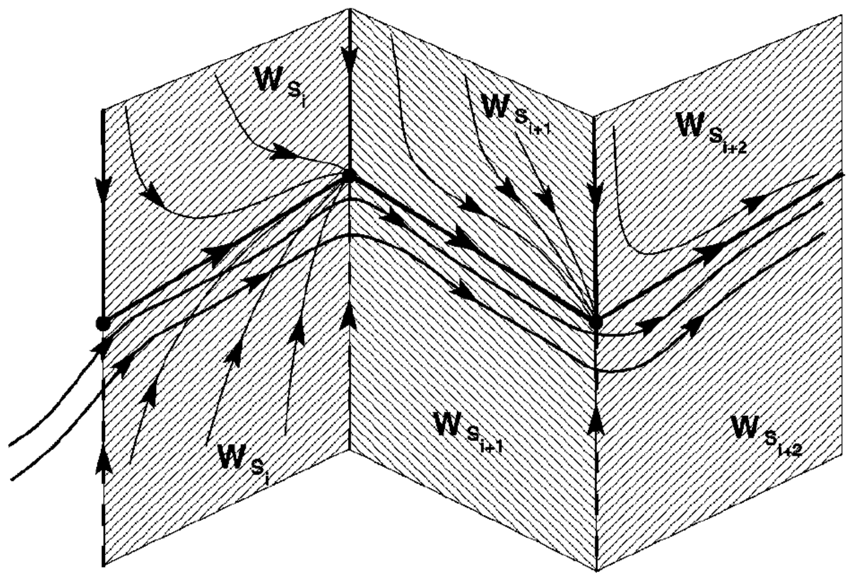
Bifurcations from maximally symmetric solutions to ones with smaller isotropy groups. eg. cluster states.

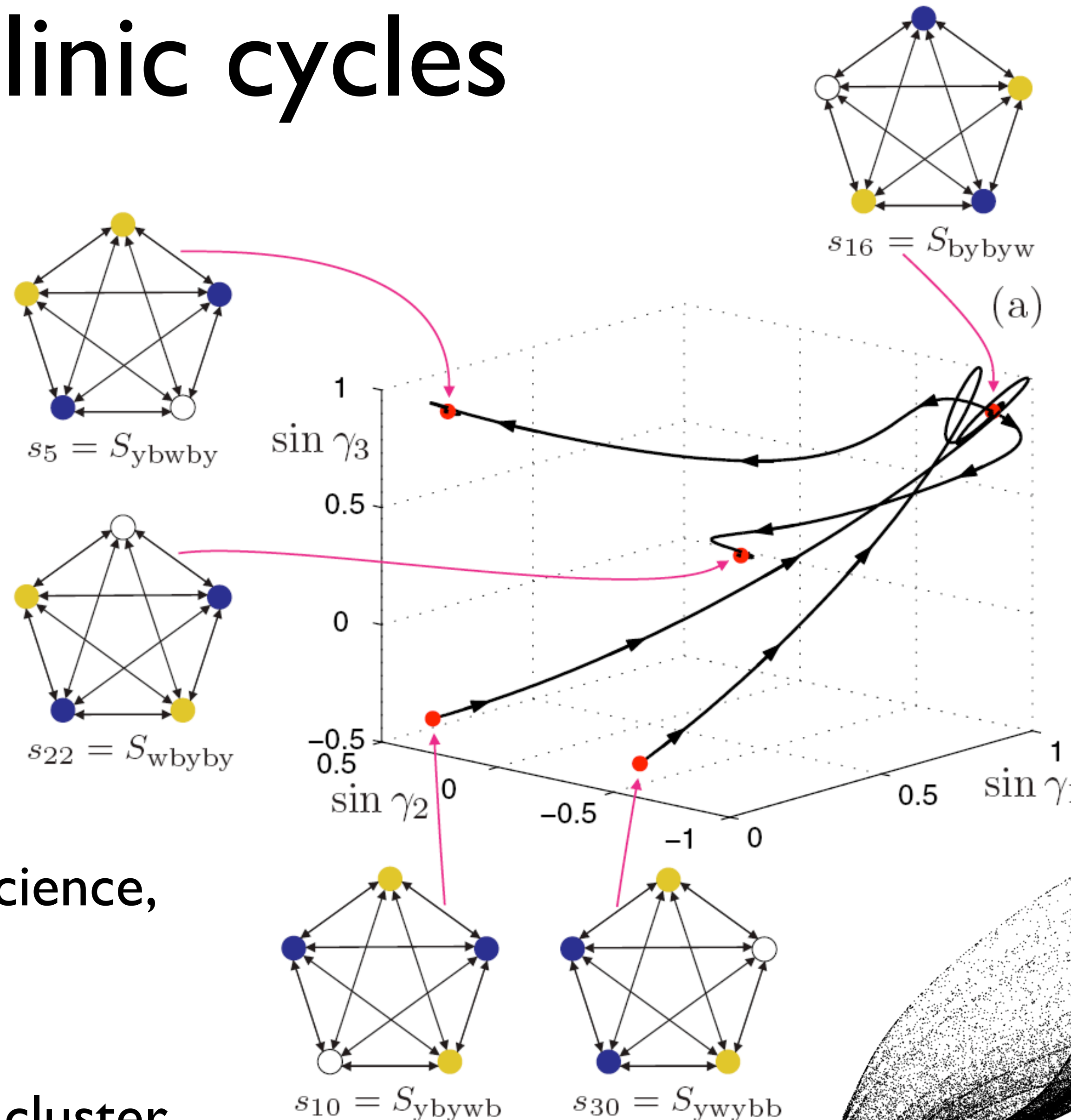
Synchrony $\lambda = -\epsilon H'(0)$ **Asynchrony** $\lambda_n = -2\pi i n \epsilon H_{-n}$

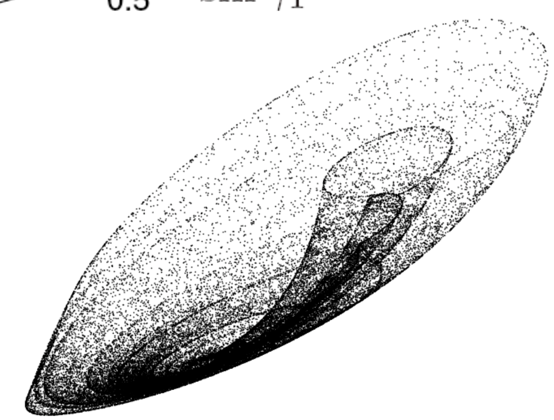


Heteroclinic cycles

Winnerless networks



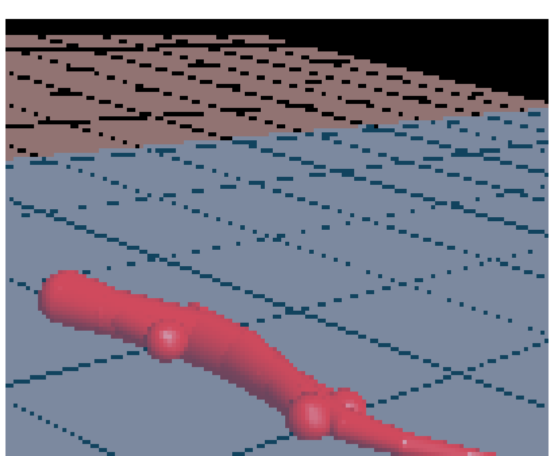


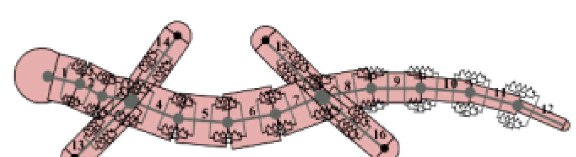


Rabinovich *et al.* Dynamical principles in neuroscience, Rev. Mod. Phys., 78, 2006.

Ashwin *et al.* SIADS, Dynamics on networks of cluster states for globally coupled phase oscillators, 6, 2007.

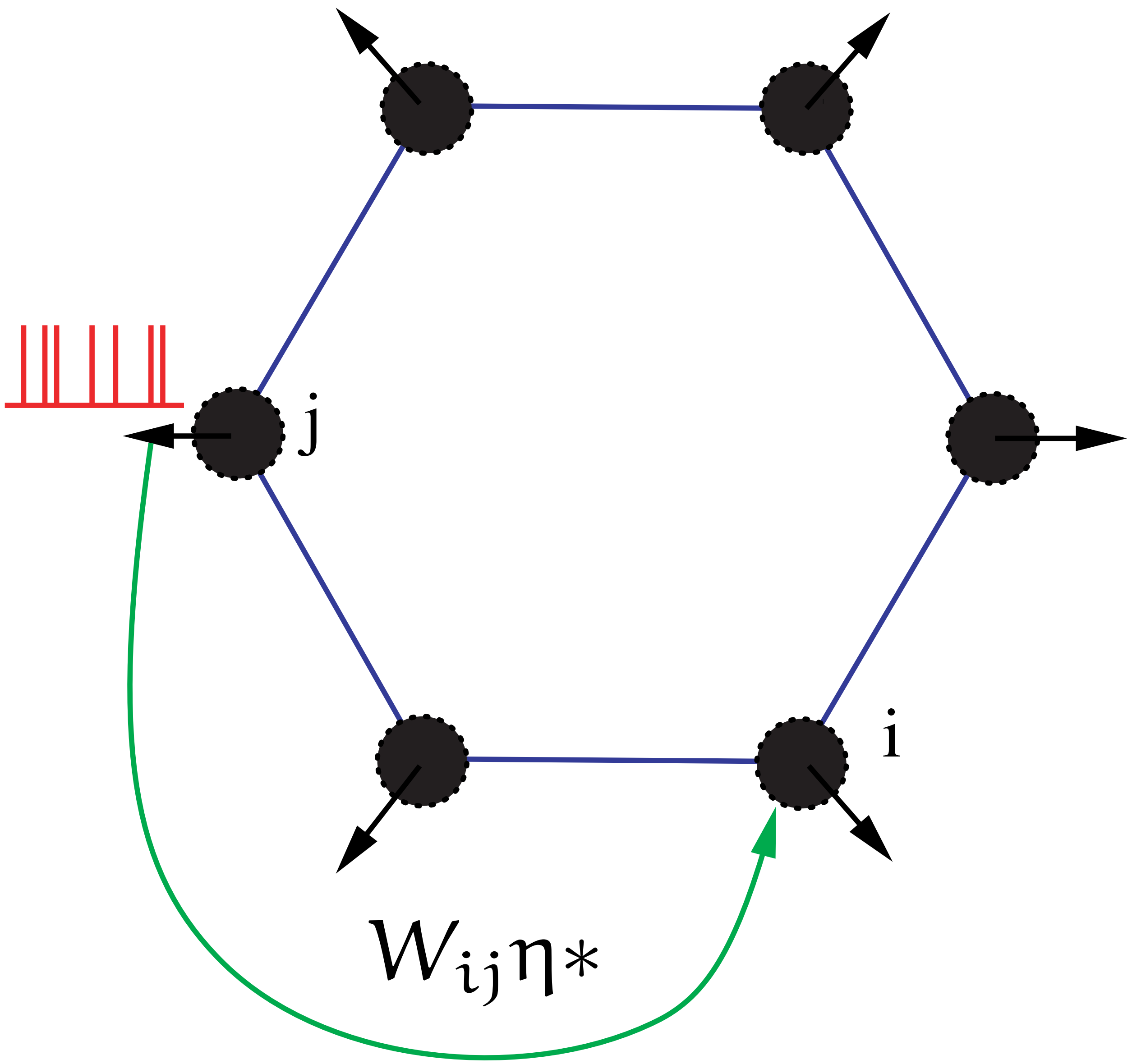
Applications of weakly coupled oscillator theory to CPGs, robot control, ...

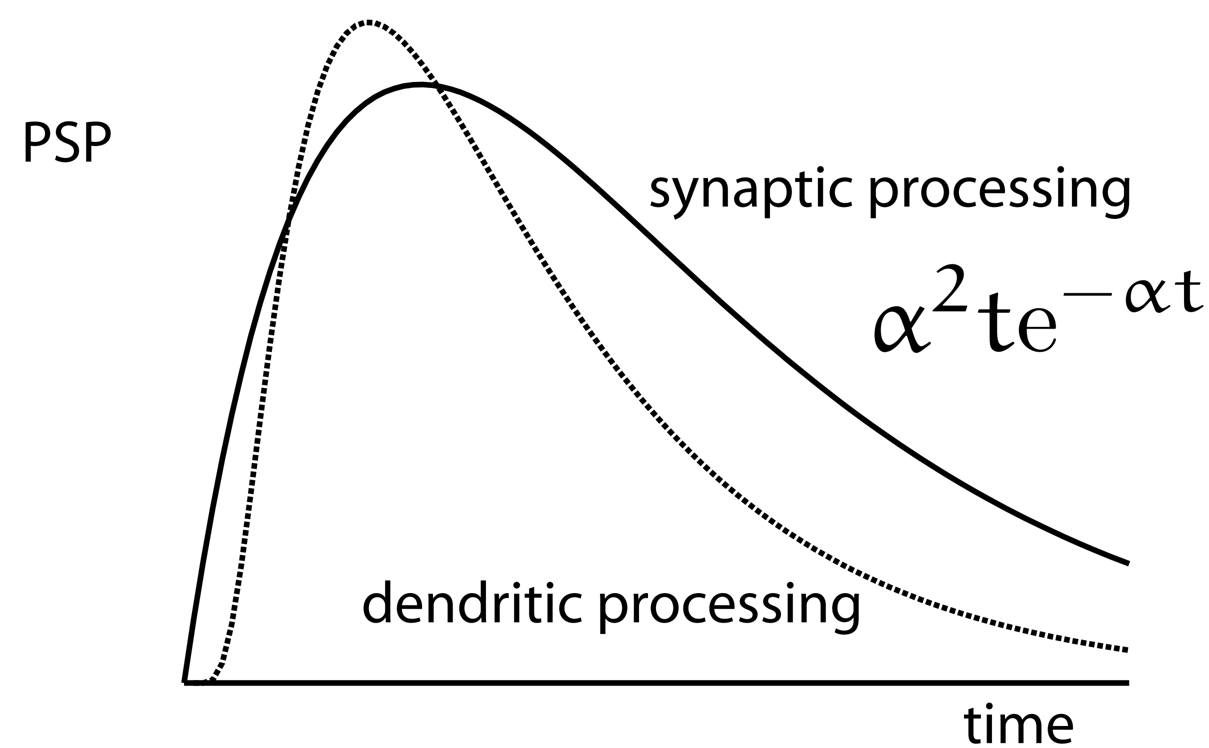




Biorobotics lab at EPFL
http://biorob.epfl.ch/

Strongly coupled synaptic networks

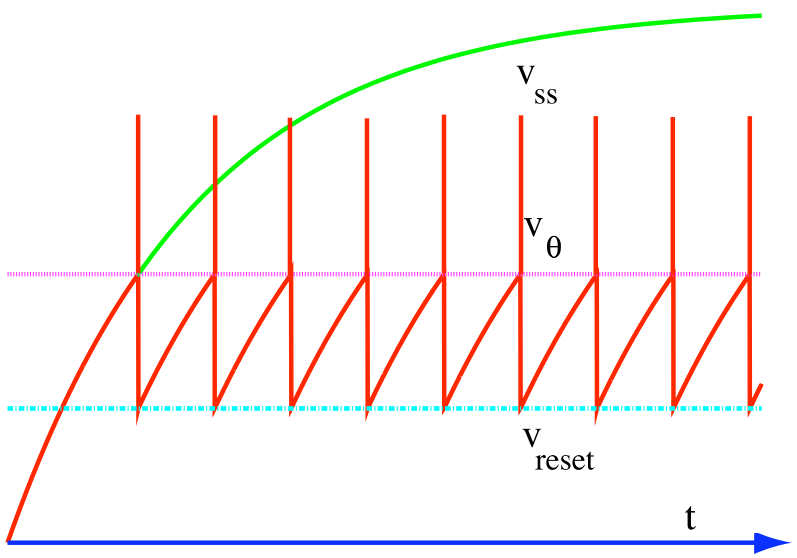




$$s_i(t) = g_s(v_s - v_i(t)) \sum_{j=1}^{N} W_{ij} \sum_{m \in \mathbb{Z}} \eta(t - T_j^m)$$

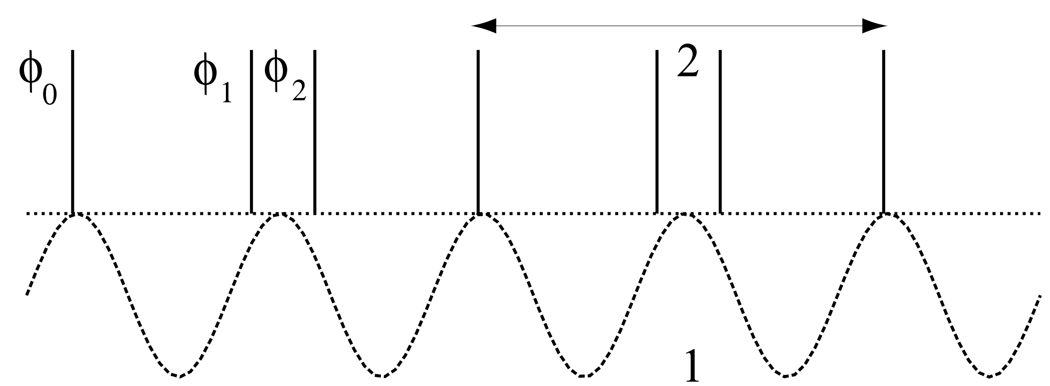
$$T_i^m = \inf\{t \mid v_i(t) > h,\ \dot{v}_i > 0,\ t > T_i^{m-1}\}$$

Integrate-and-fire neurons



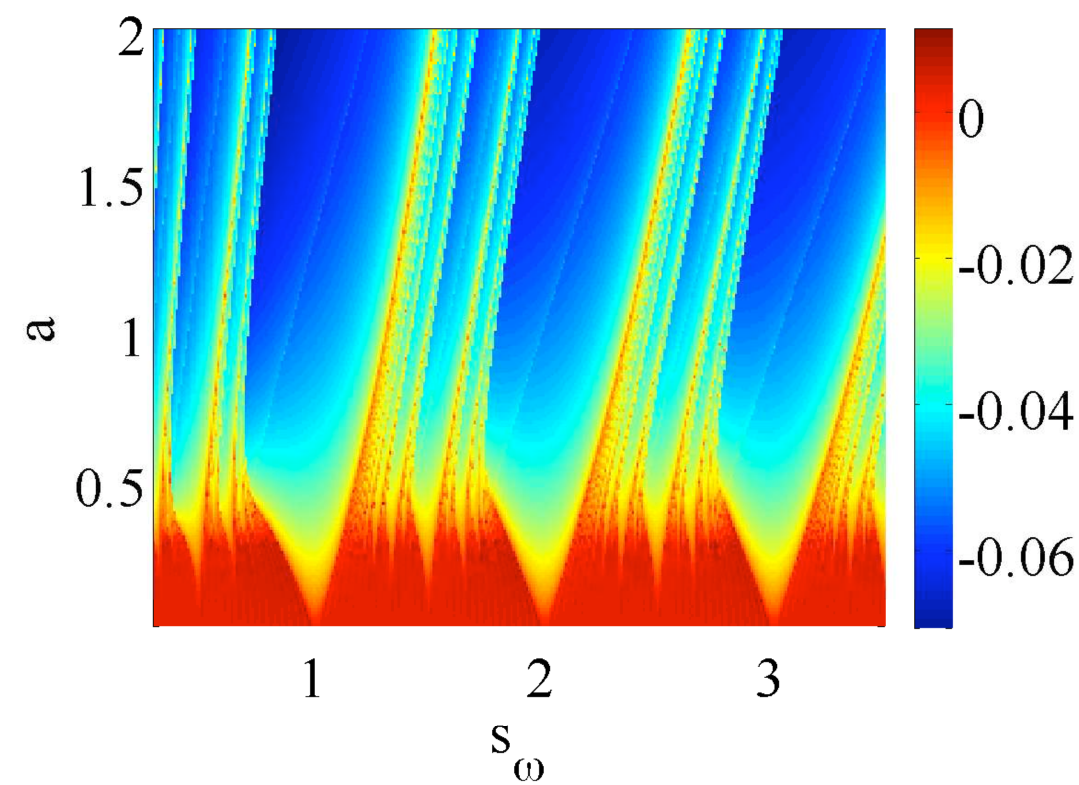
$$\frac{dv}{dt} = -\frac{v}{\tau} + A(t), \qquad t \in (T^m, T^{m+1})$$

subject to nonlinear reset



Periodic forcing gives $p : q$ mode-locked states

Implicit map of firing times

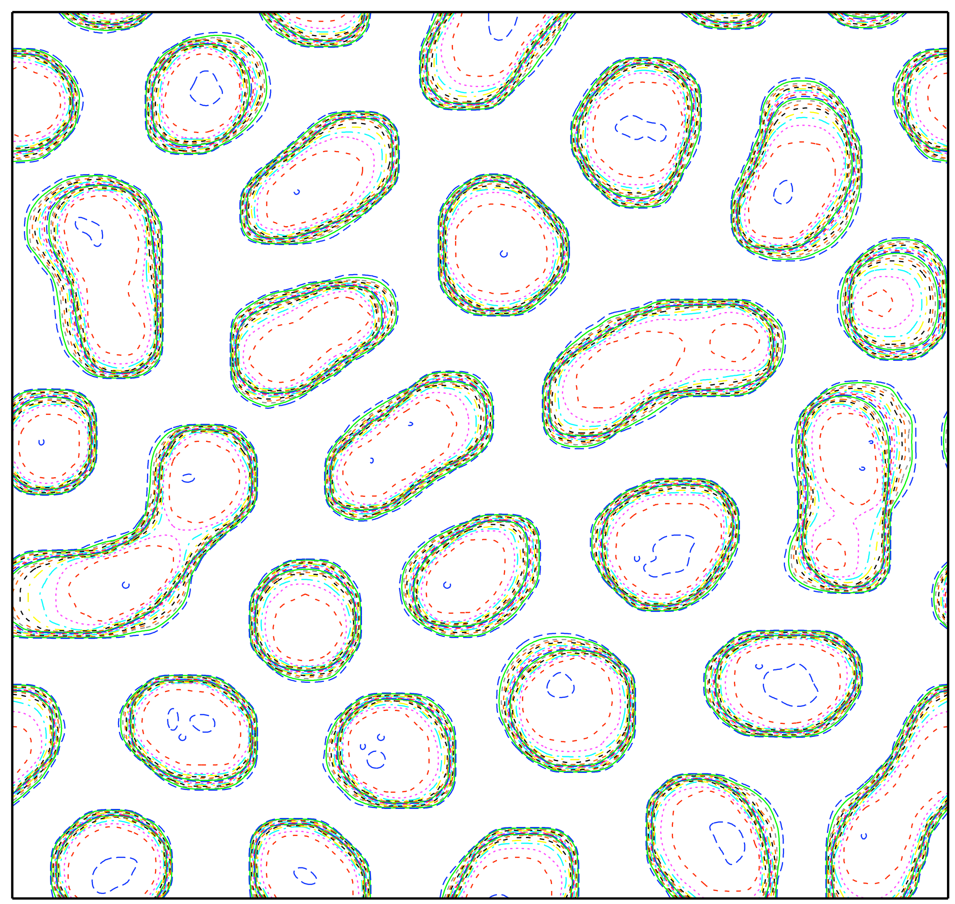


Arnol'd tongue structure dominated by non-smooth bifurcations

CML - discrete time IF

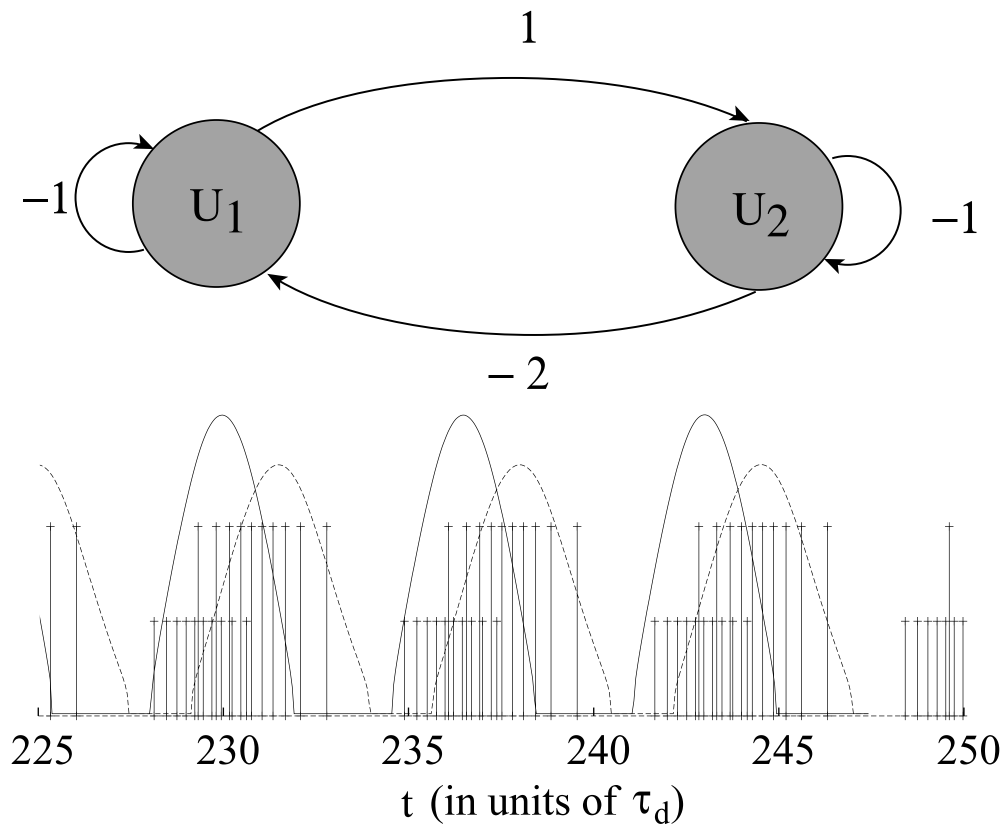
$$V_i(n+1) = [\gamma V_i(n) + \epsilon \sum_j W_{ij} a_j(n)]\Theta(1 - V_i(n))$$

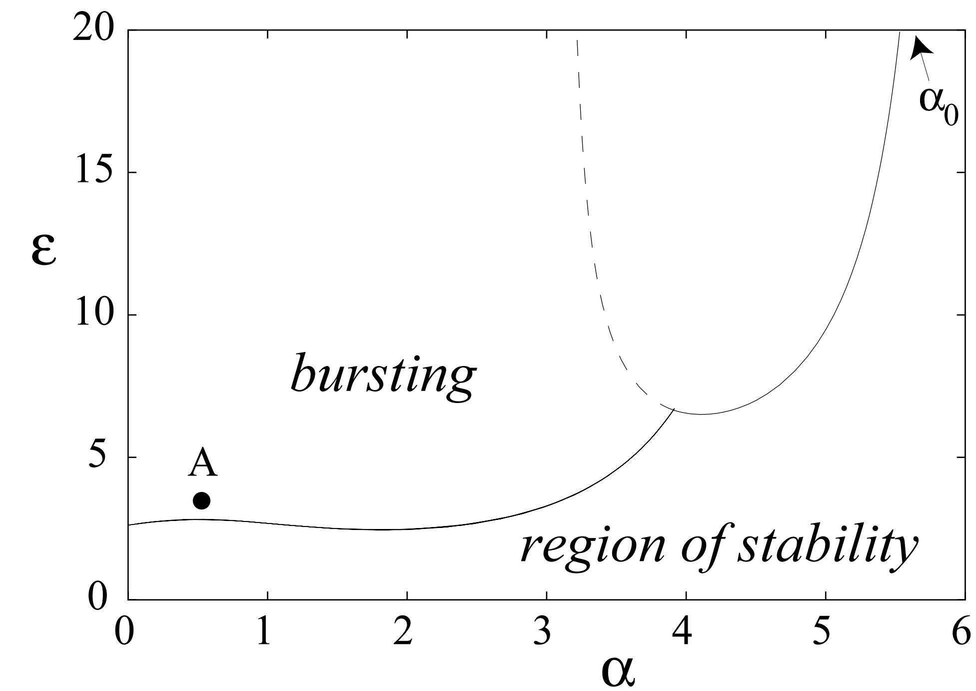
$$a_i(n) = \Theta(V_i(n) - 1)$$



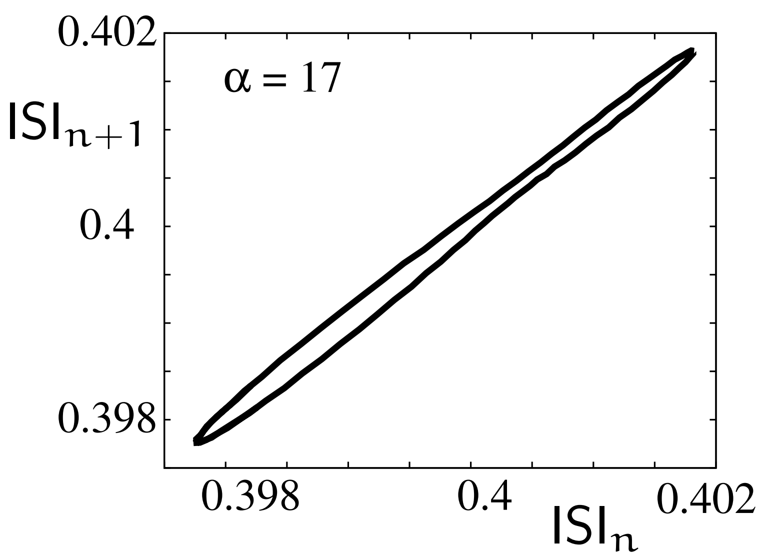
Mexican hat interaction

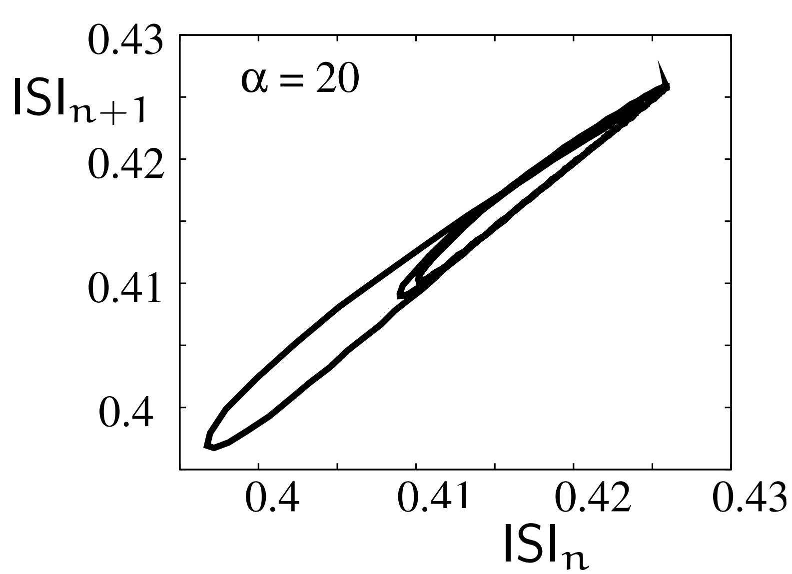
Network firing maps



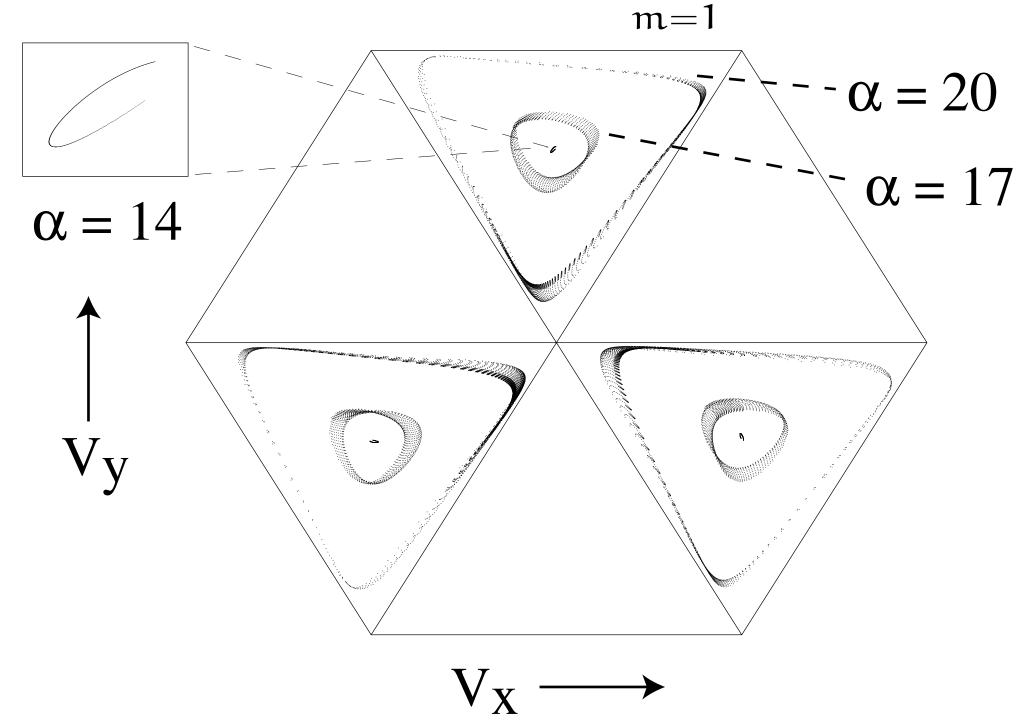


Global heteroclinic bifurcation (N=3) $\mathrm{ISI}_n = T^{n+1} - T^n$



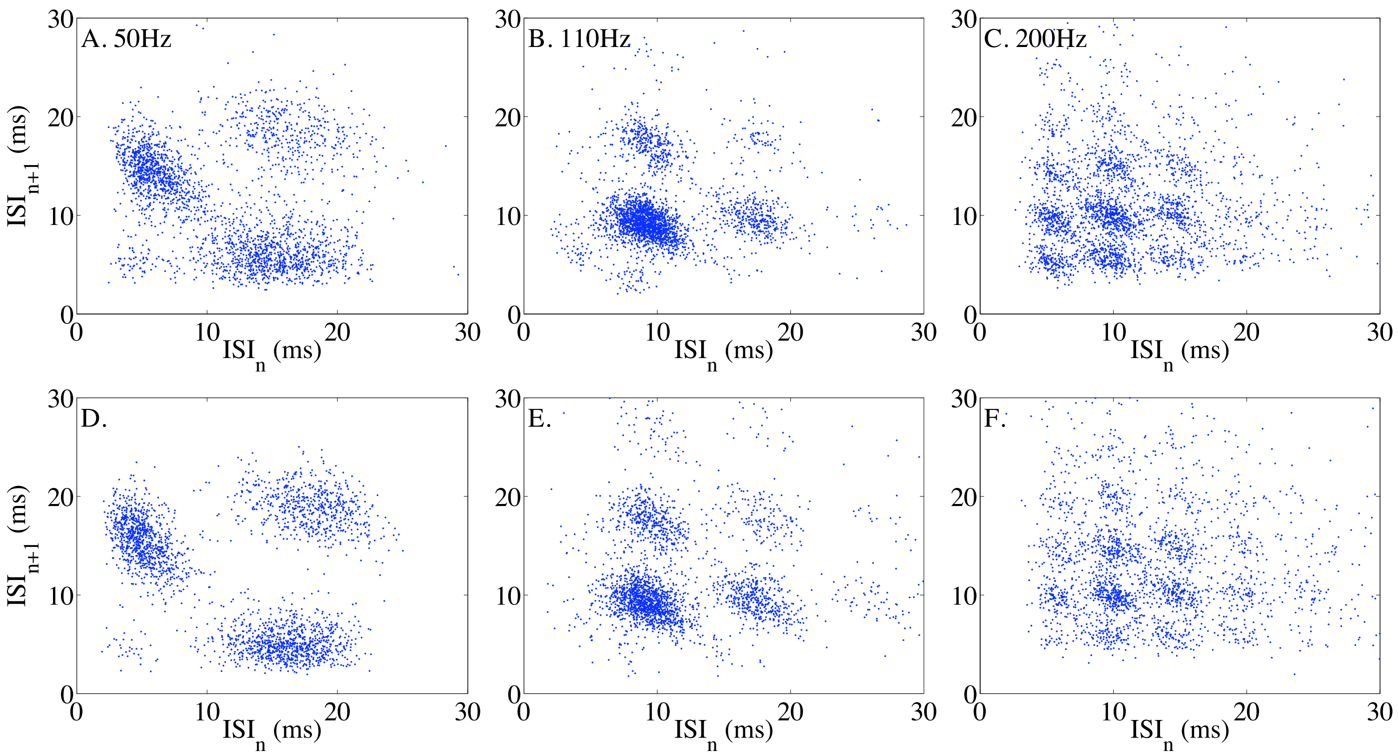


$$V_x(t) + iV_y(t) = \sum_{m=1}^{3} e^{2\pi i m/3} U_m(t)$$



P C Bressloff and S Coombes 2000 Dynamics of strongly-coupled spiking neurons, Neural Computation, Vol 12, 91-129

Fits to data

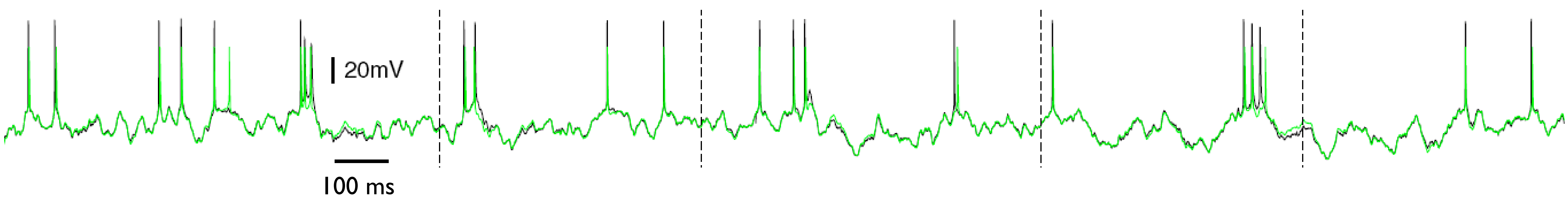


VCN stellate cell

Linear IF and threshold noise

$$\mathrm{ISI}_n = T^{n+1} - T^n$$

J Laudanski *et al.*, Journal of Neurophysiology, 103, 2010



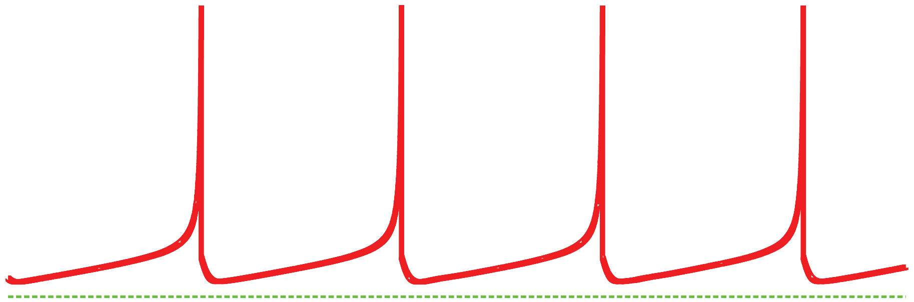
Nonlinear IF

$$-\frac{1}{\tau}(v - v_L) + \frac{\kappa}{\tau}e^{(v-v_\kappa)/\kappa}$$

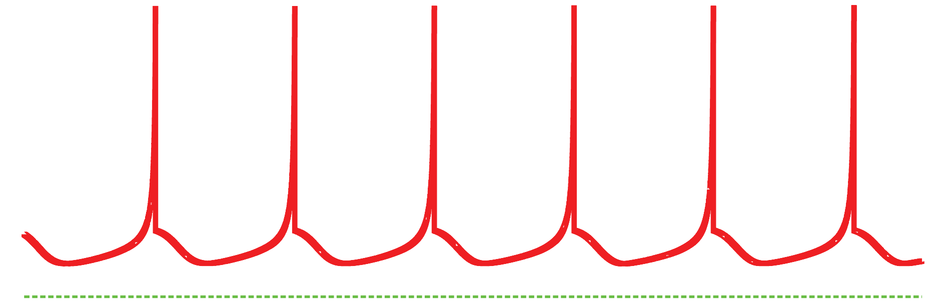
Layer V cortical pyramidal cell

Badel *et al.*, Journal of Neurophysiology, 99, 2010

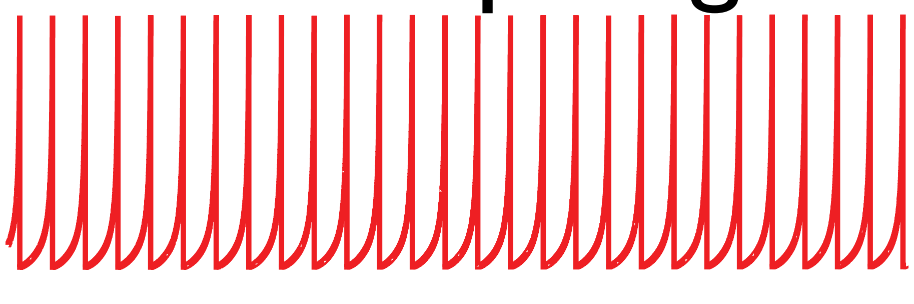
Regular spiking



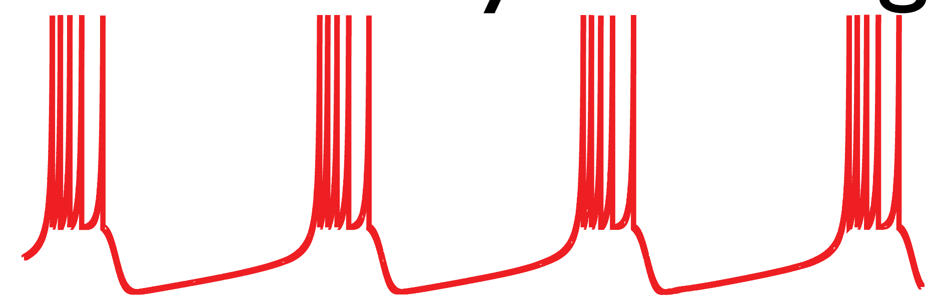
Chattering

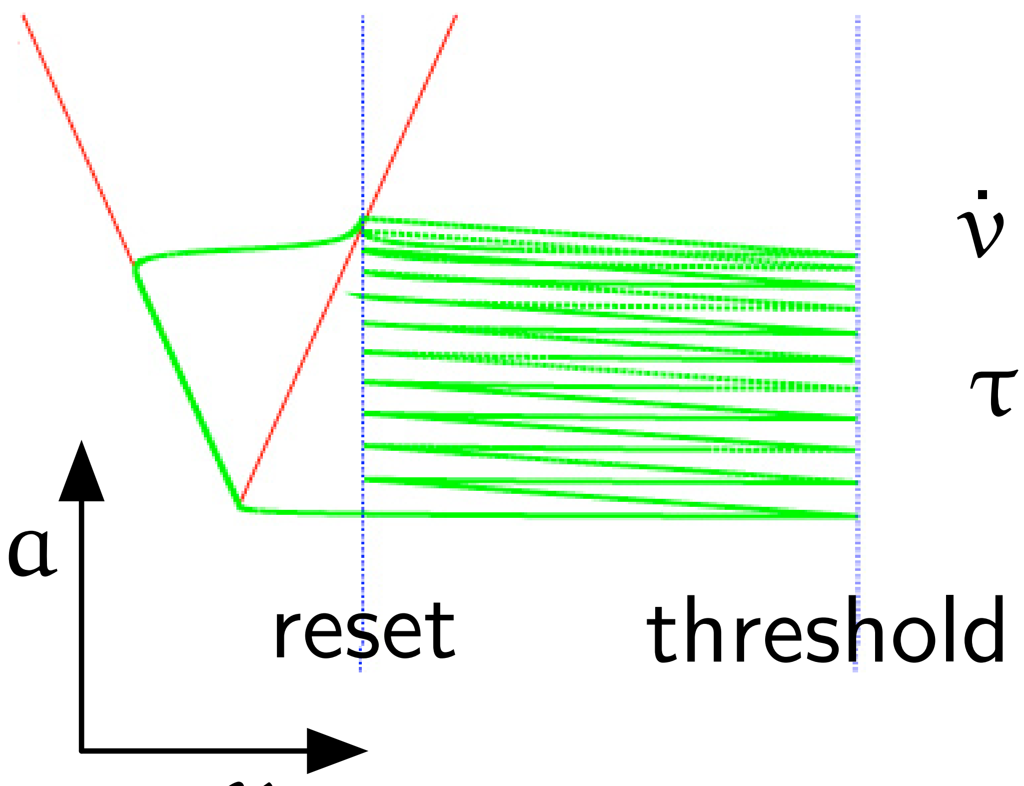


Fast spiking



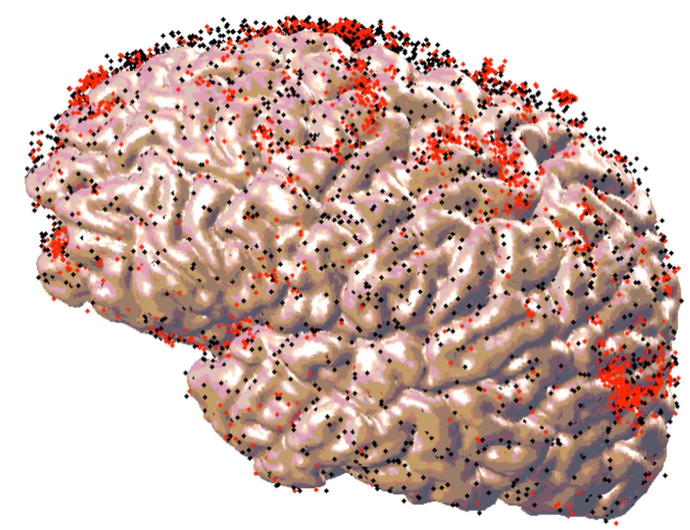
Intrinsically bursting





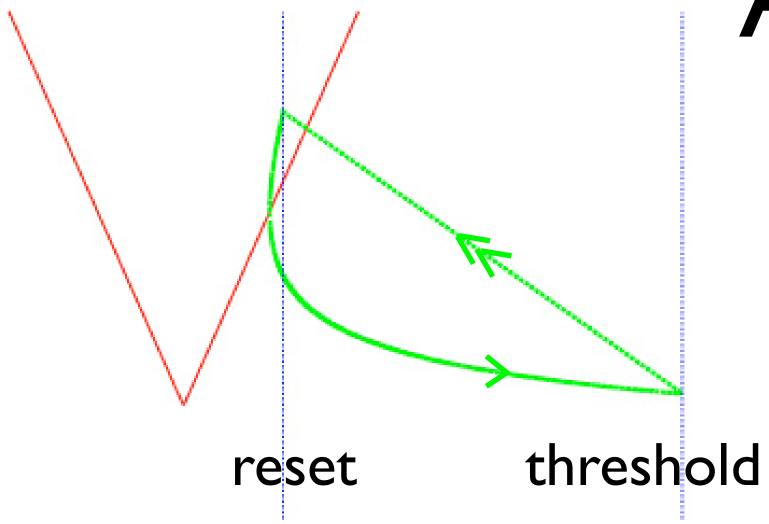
$$\dot{v} = |v| +$$

$$\tau \dot{a} = -a$$



S Coombes and M Zachariou 2009, in Coherent Behavior in Neuronal Networks (Ed. Rubin, Josic, Matias, Romo), Springer.

Absolute IF networks



$$a(T^m) \to a(T^m) + g_a/\tau_a$$

Orbit and PRC in closed form (pwl system)

Gap jn network: asynchronous state

network averages

~

time averages

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} v(t + jT/N) = \frac{1}{T}\int_0^T v(t)dt \equiv v_0$$

$$\dot{v} = |v| - \epsilon v + I - a + \epsilon v_0, \qquad \dot{a} = -a/\tau_a$$

advanced-retarded ode - self-consistent periodic solution

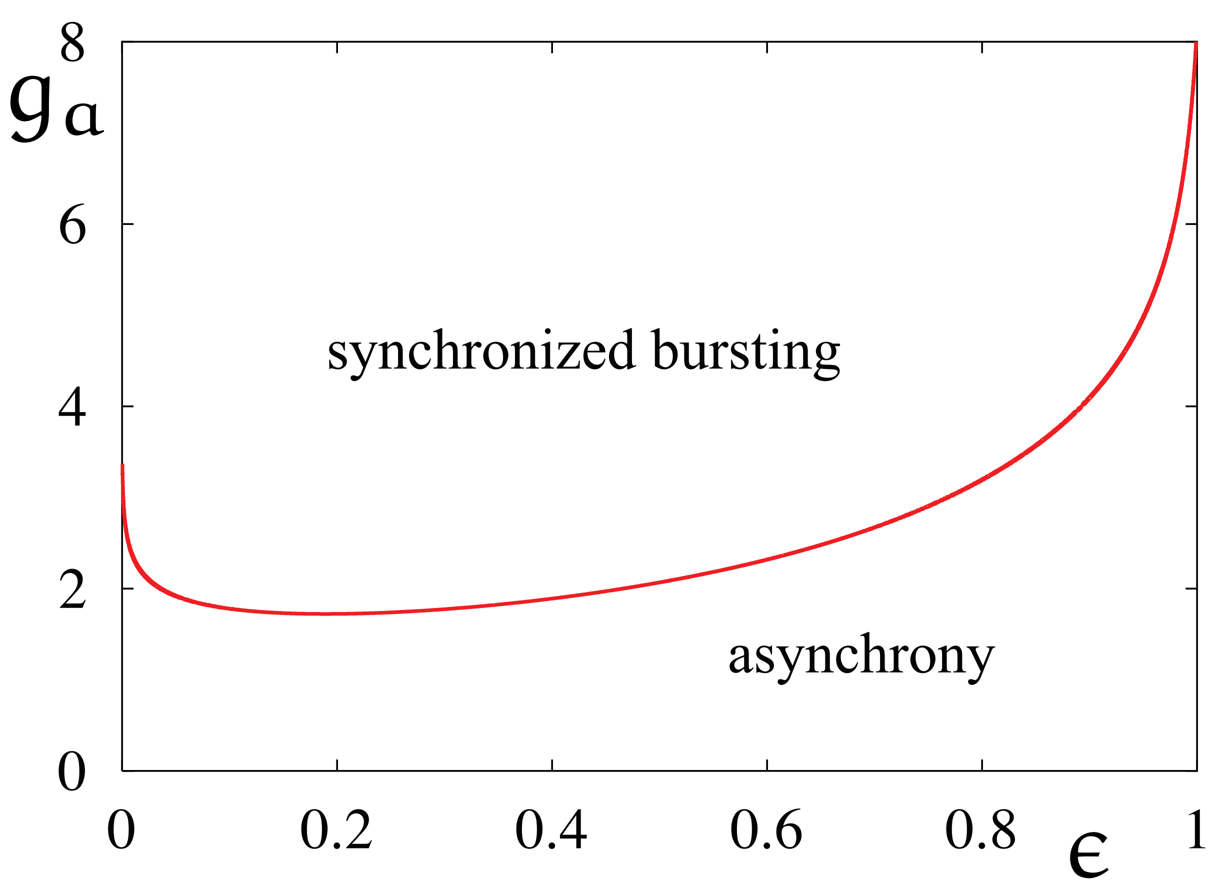
Stability and bifurcations

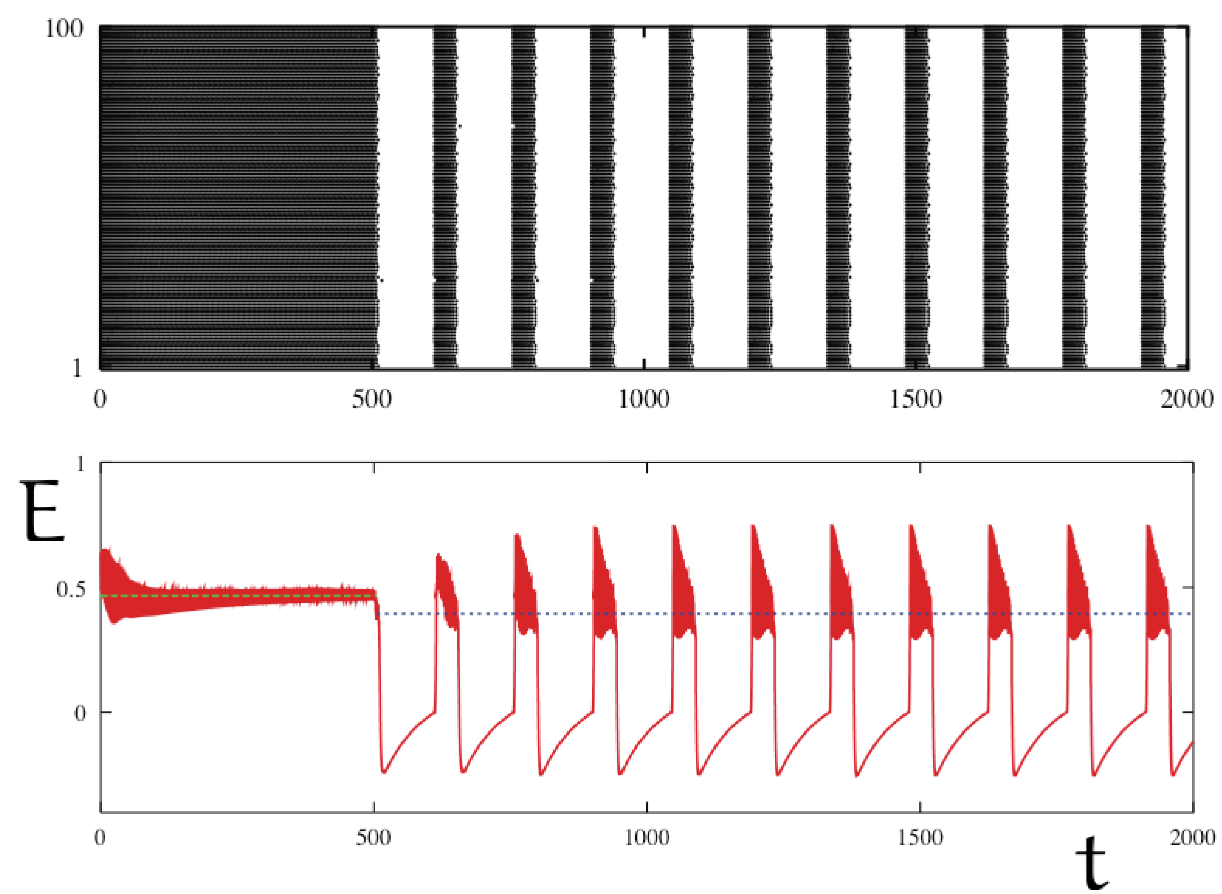
e-values as zeros of (using phase density formalism):

PRC of splay

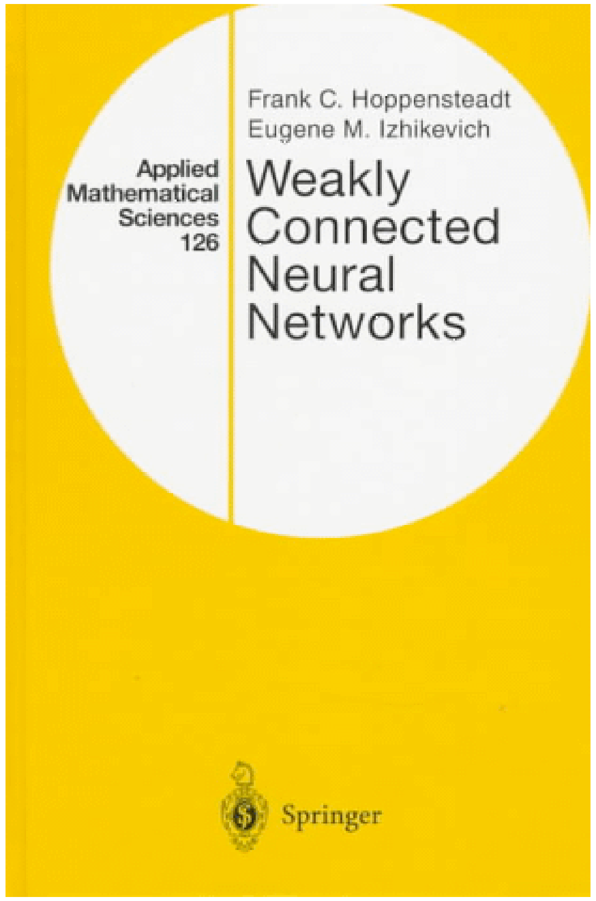
$$\mathcal{E}(\lambda) = \frac{e^{\lambda T}}{\widetilde{\nu}(\lambda)} + \epsilon\lambda T \int_0^1 R(\theta) e^{\lambda\theta T} d\theta$$

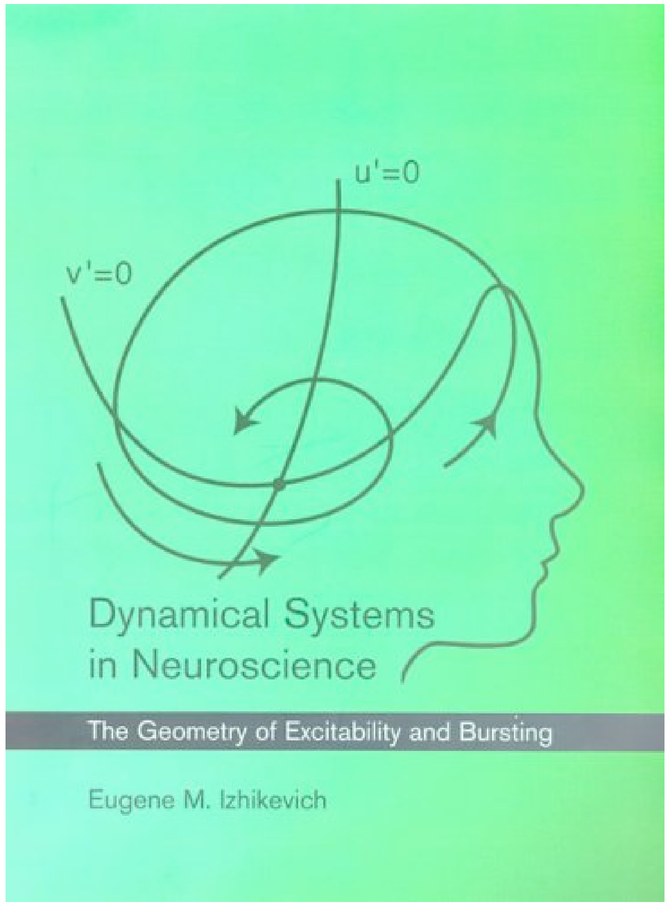
LT of orbit

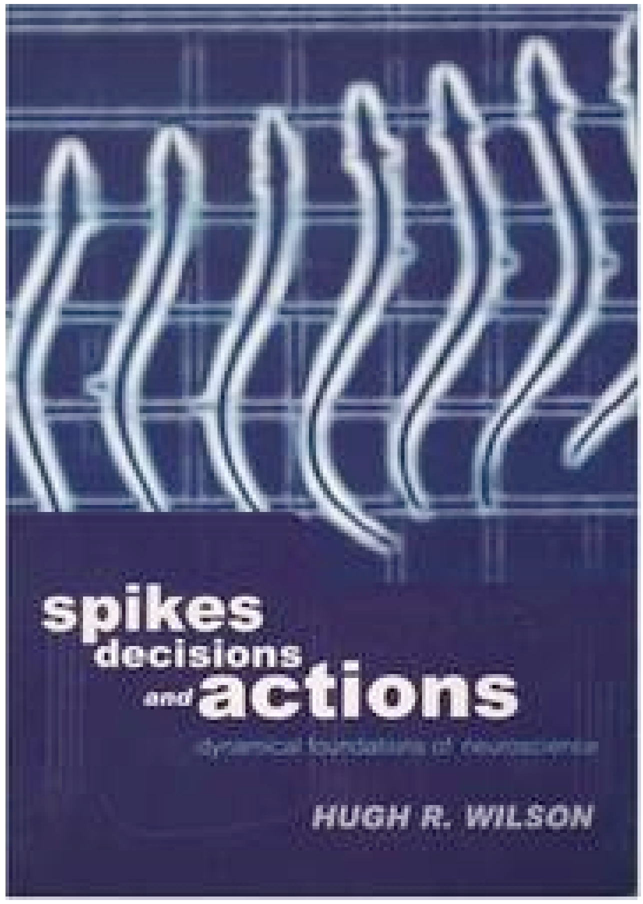


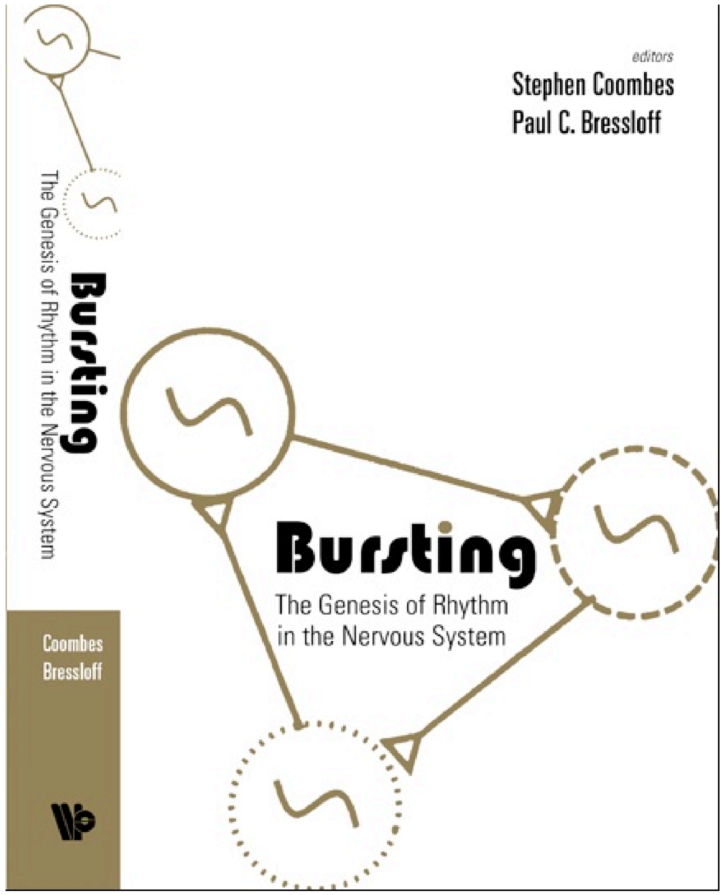


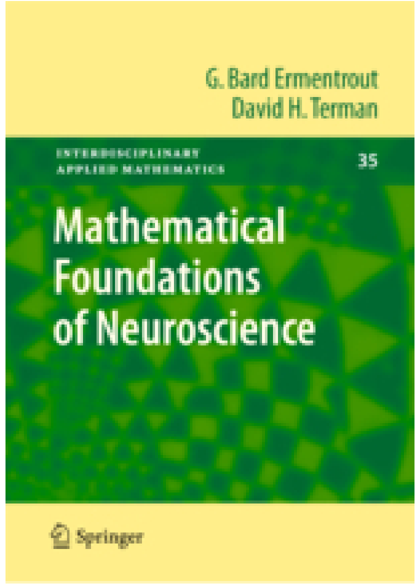
Books

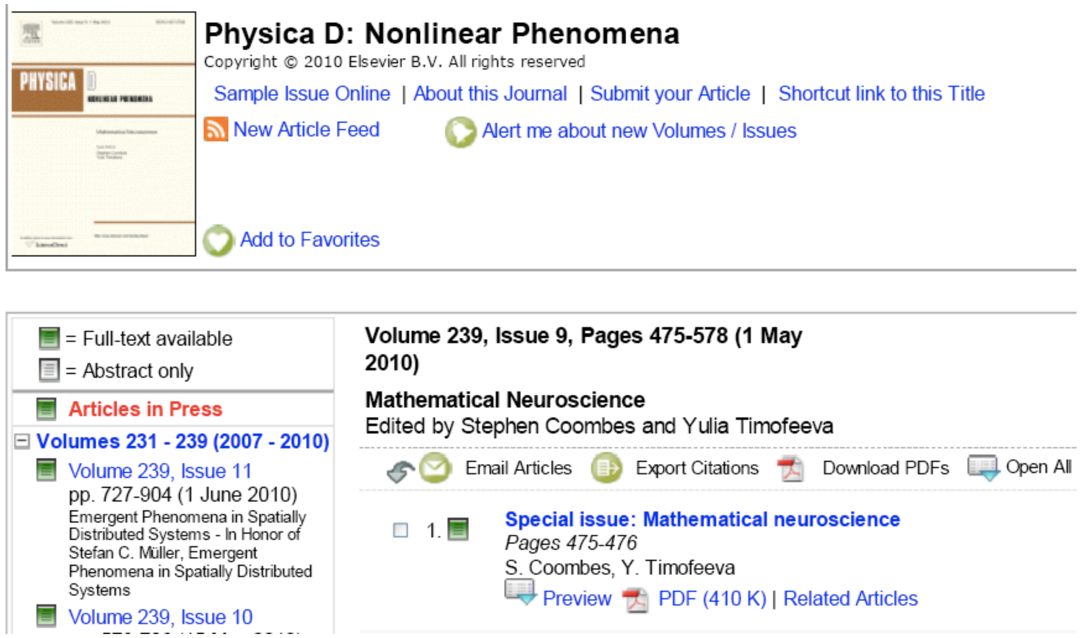






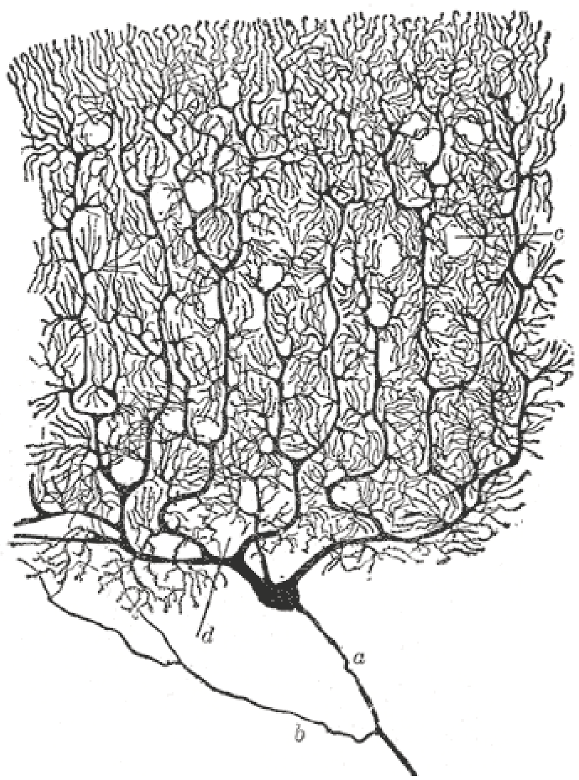






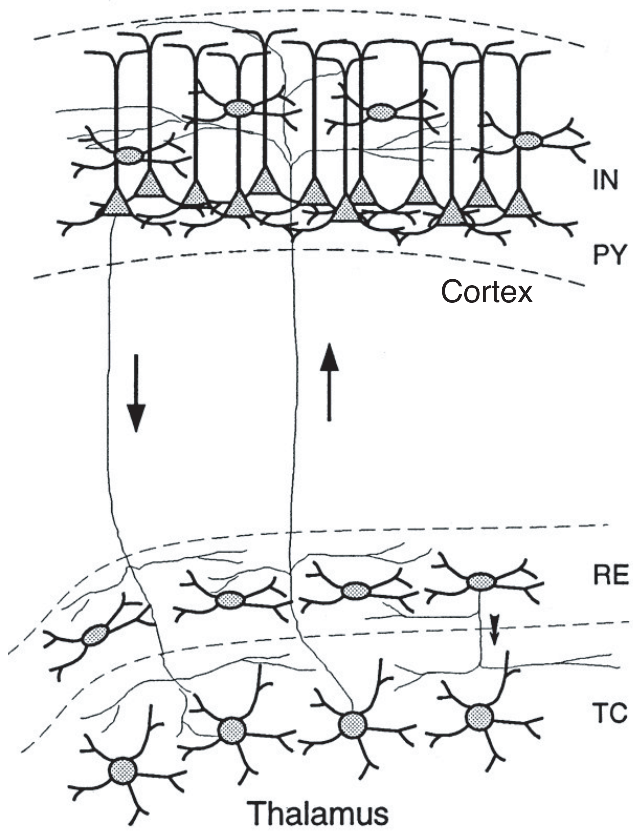
Physica D Special Issue
Mathematical Neuroscience
Vol 239, May 2010

Mathematical Neuroscience: from neurons to networks



Part II

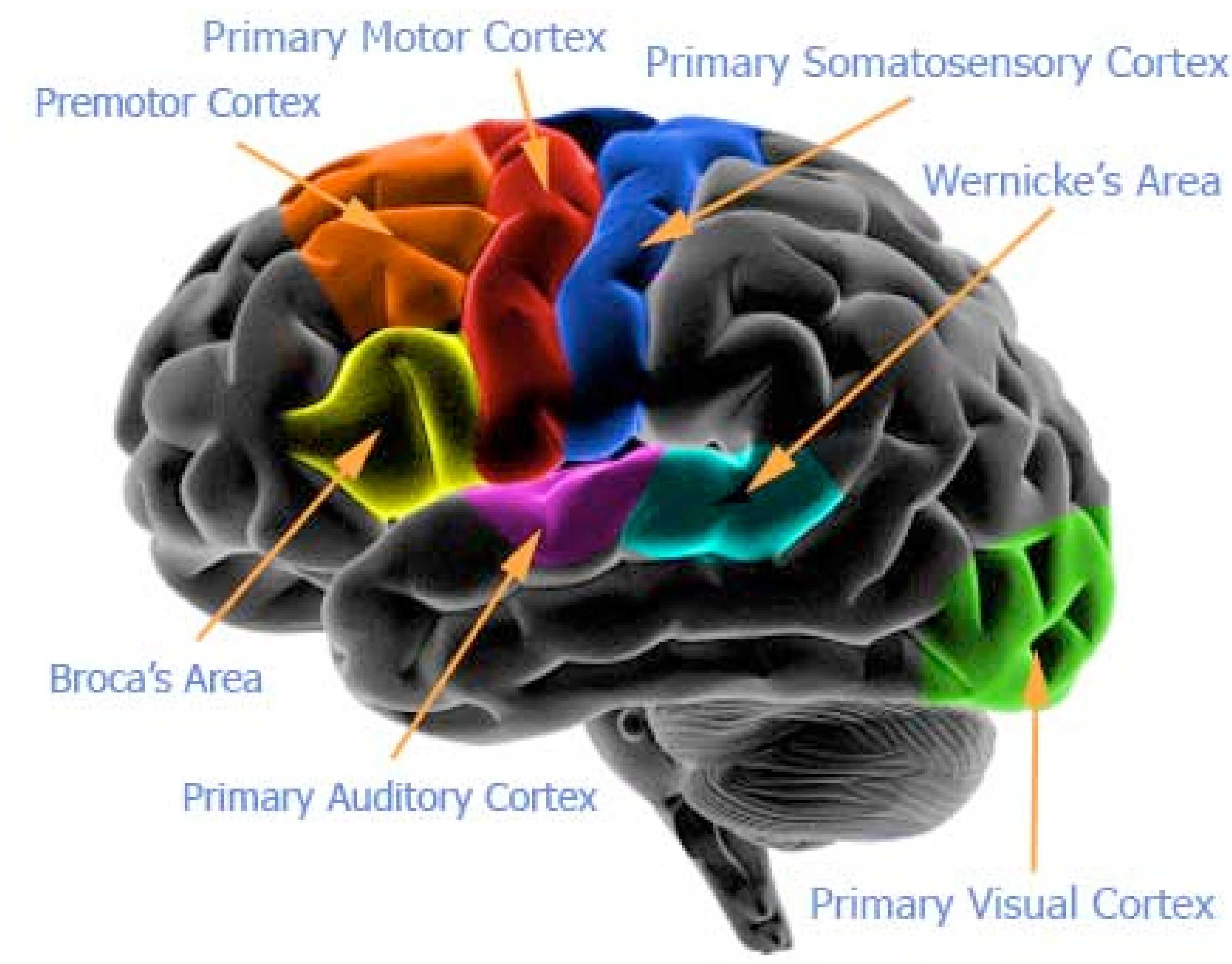
Steve Coombes

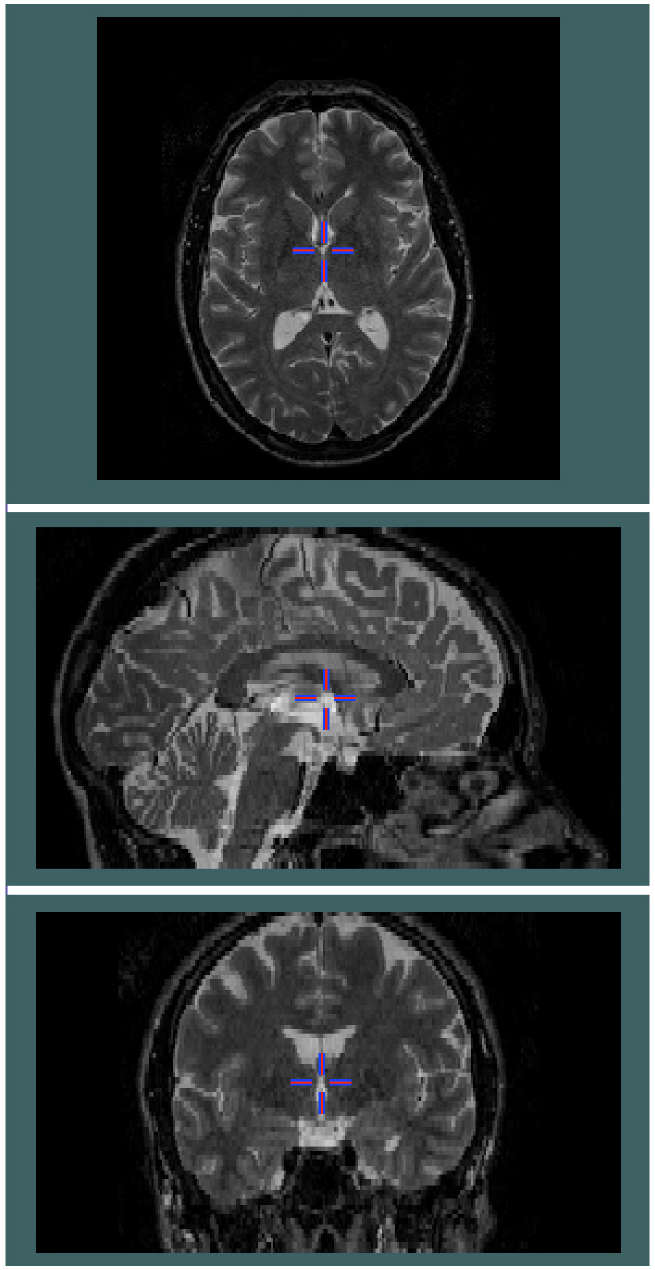


The University of Nottingham

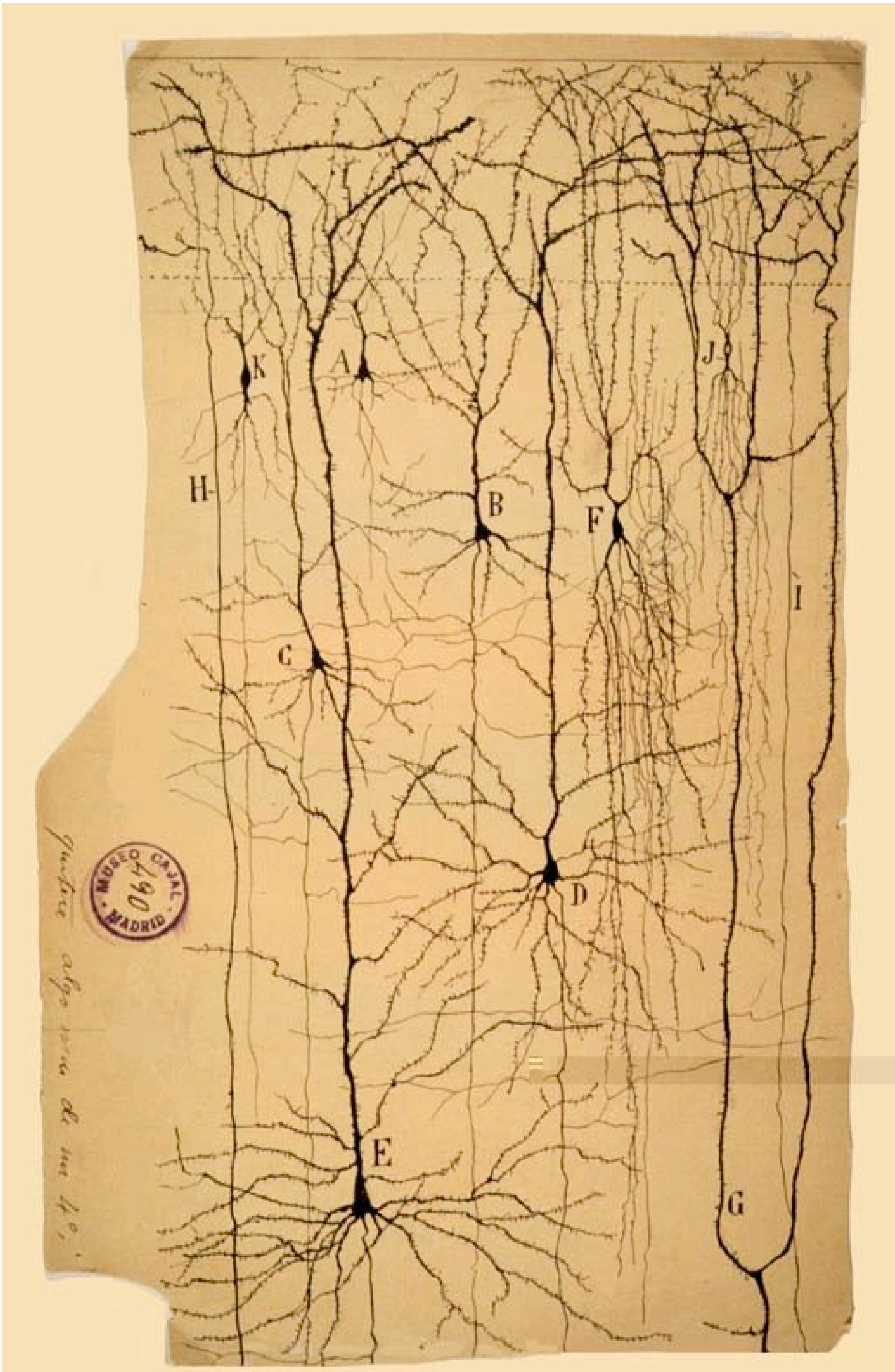
School of Mathematical Sciences

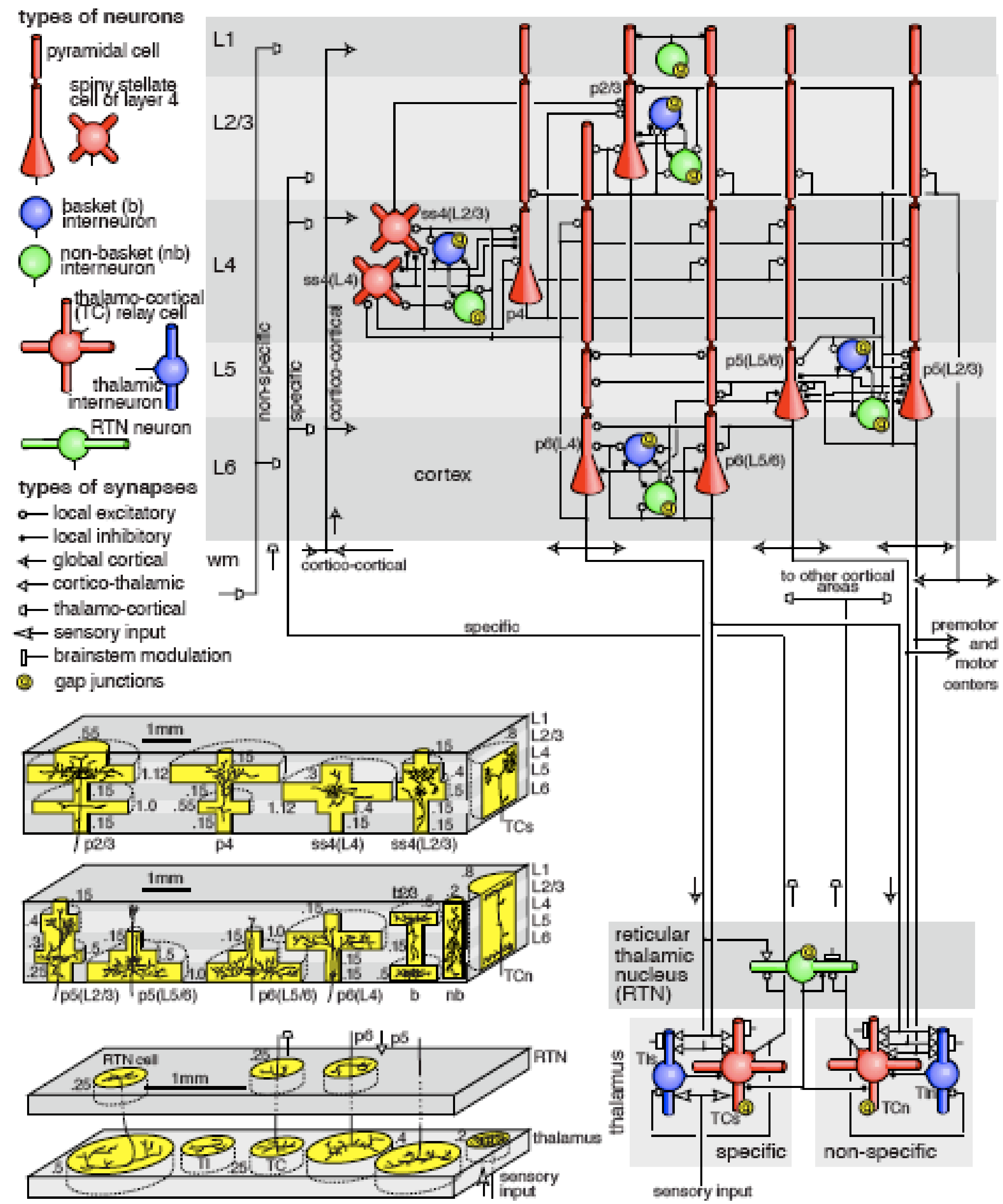
Brain and Cortex





Principal cells and interneurons

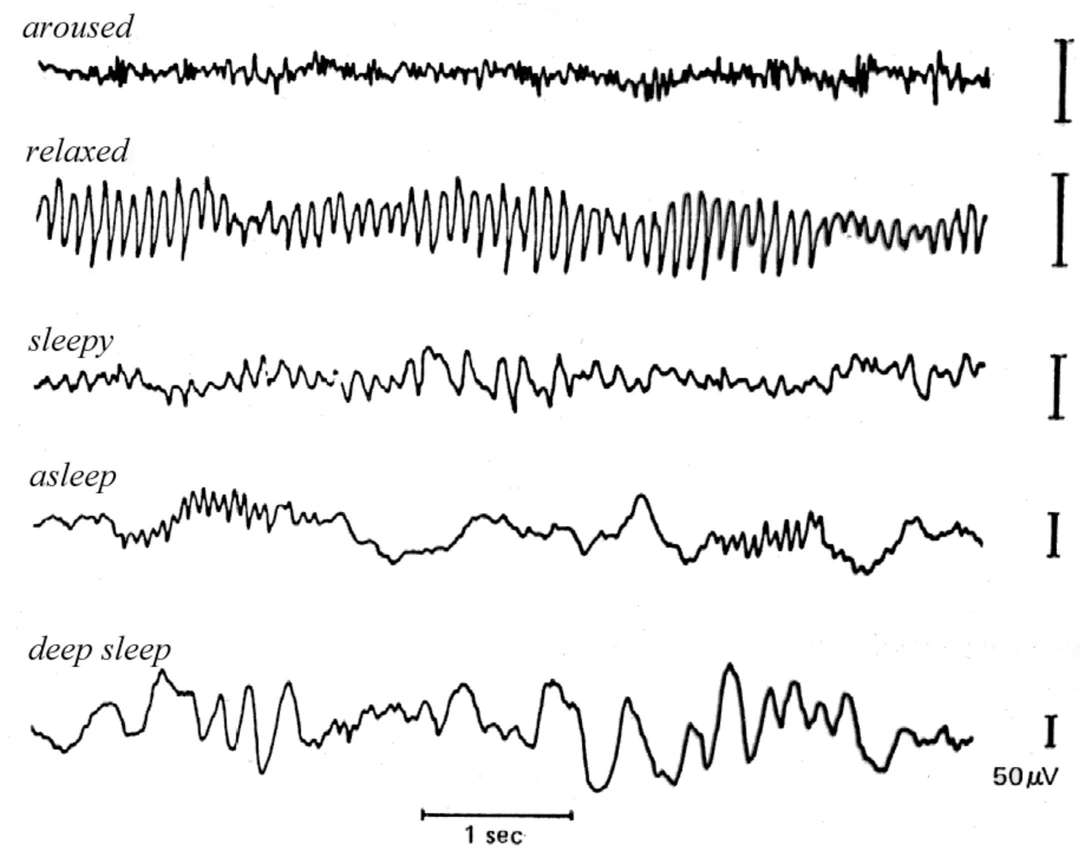




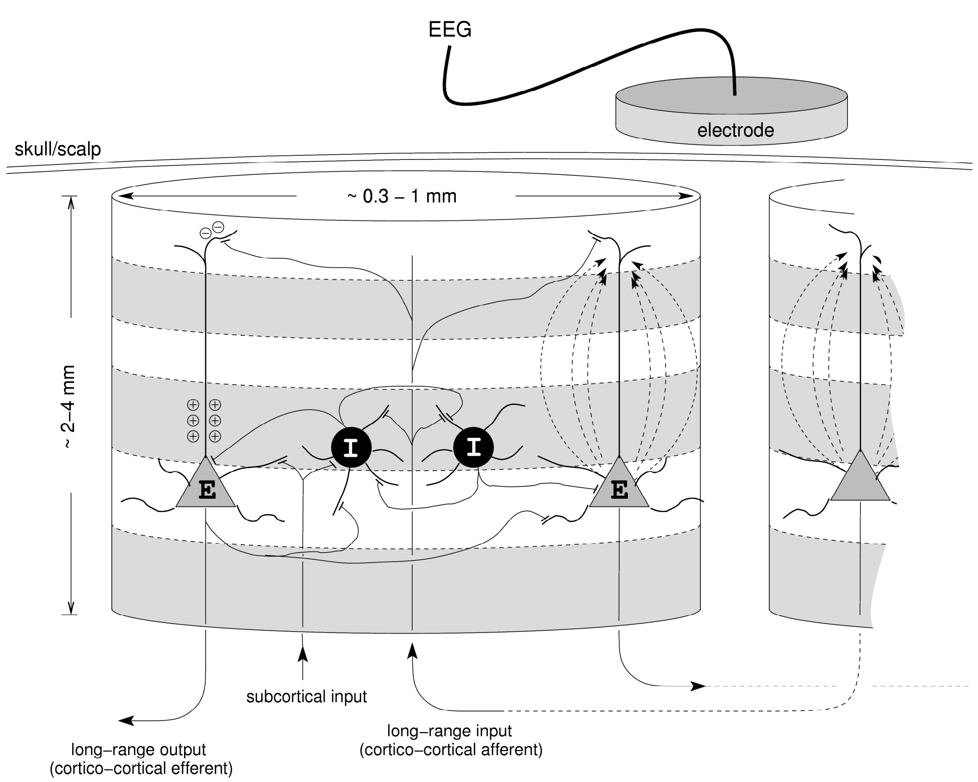
Santiago Ramón y Cajal
1900

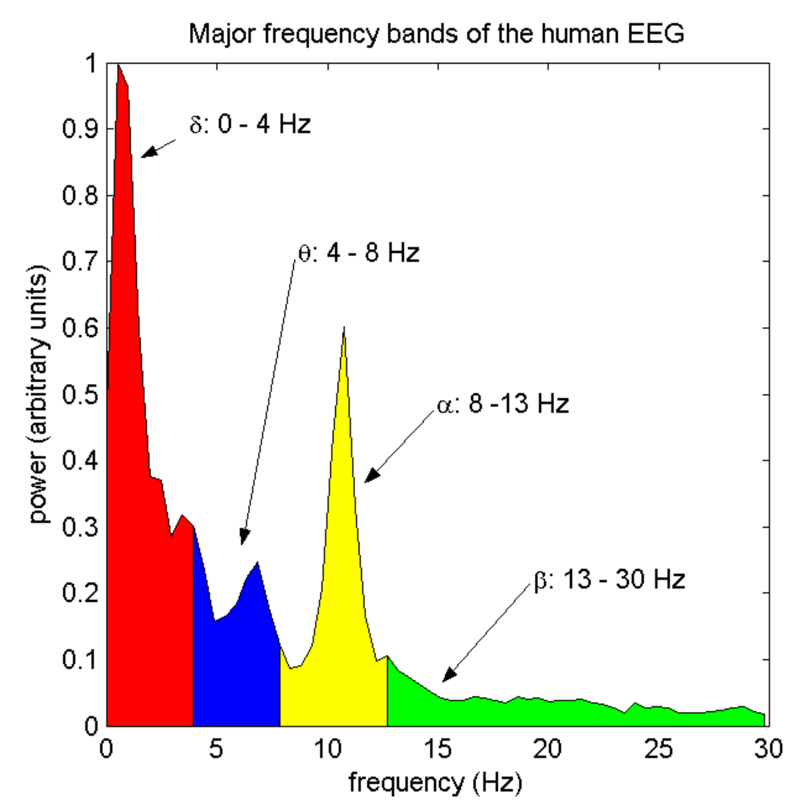
Eugene Izhikevich
2008

Electroencephalogram (EEG) power spectrum



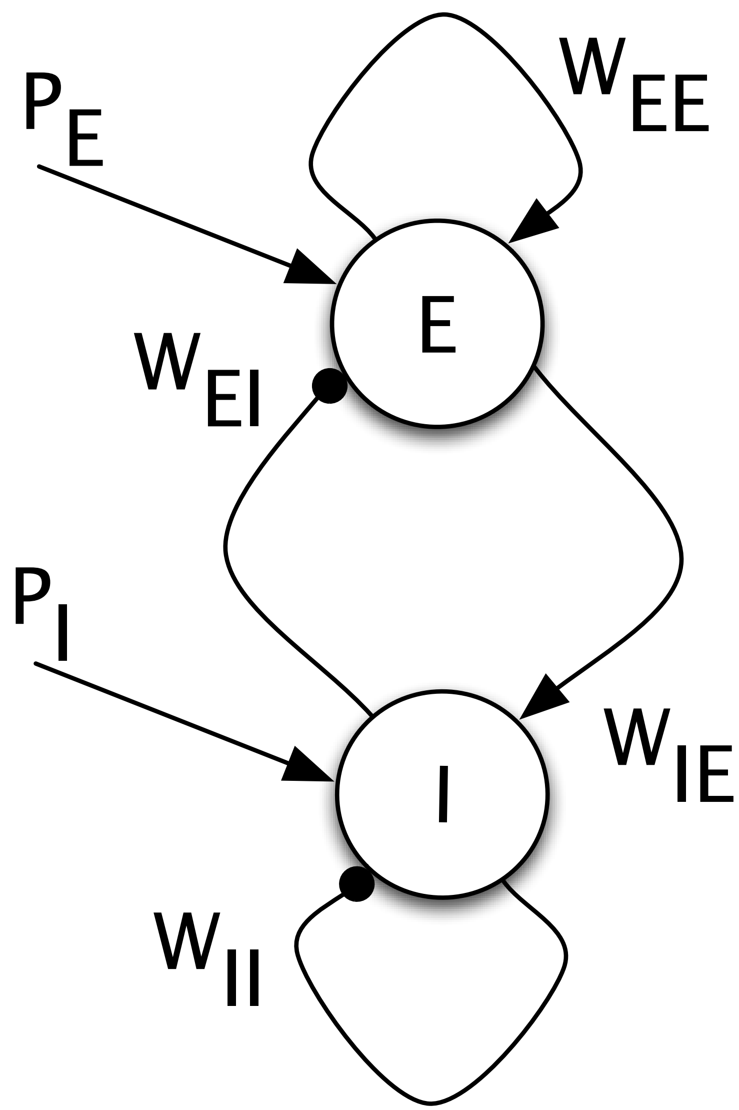
α



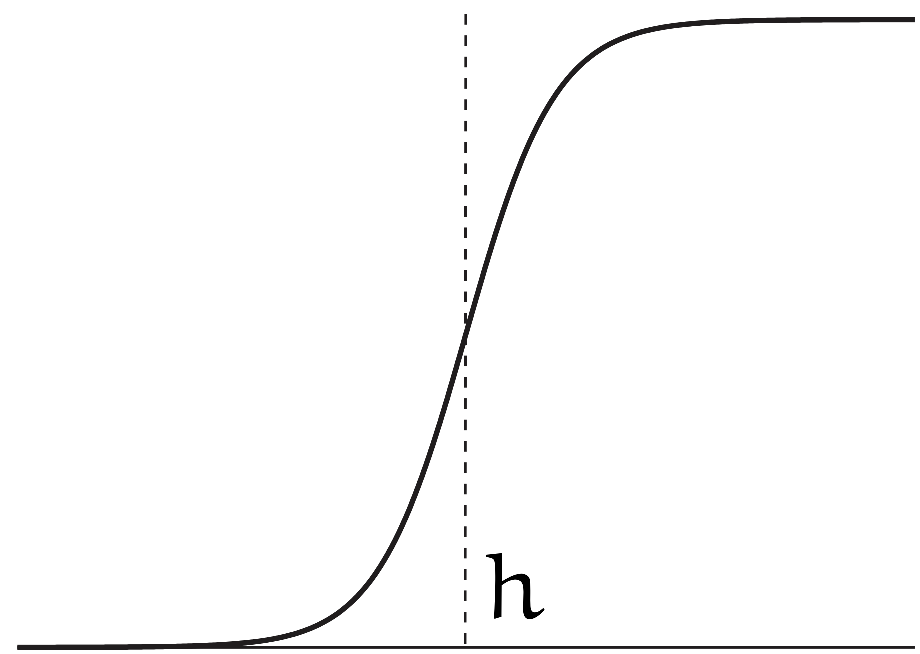


EEG records the activity of ~ 10^6 pyramidal neurons.

Population model



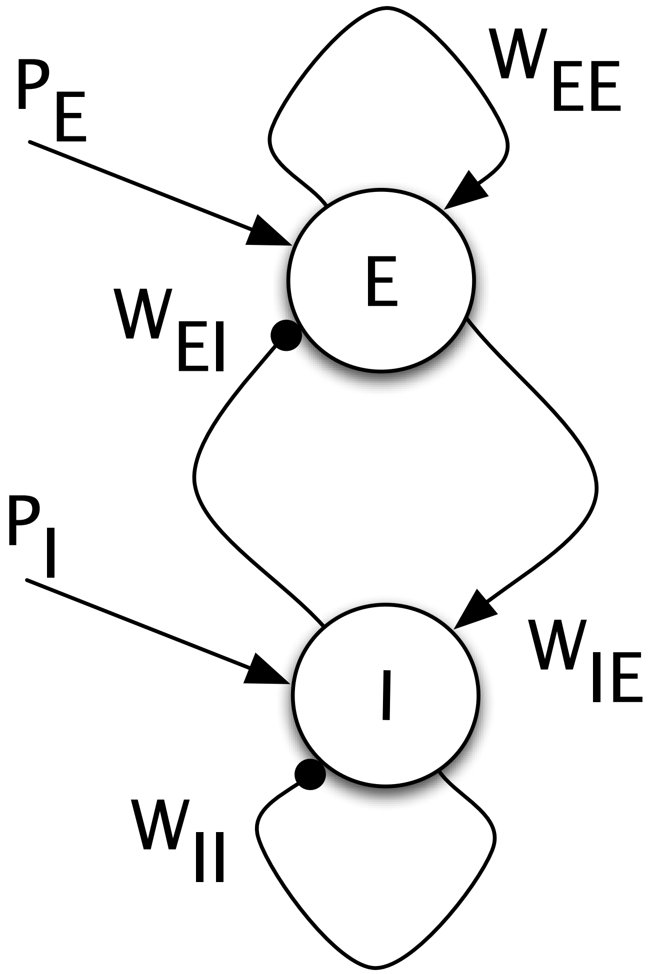
Firing rate activity $f(E)$

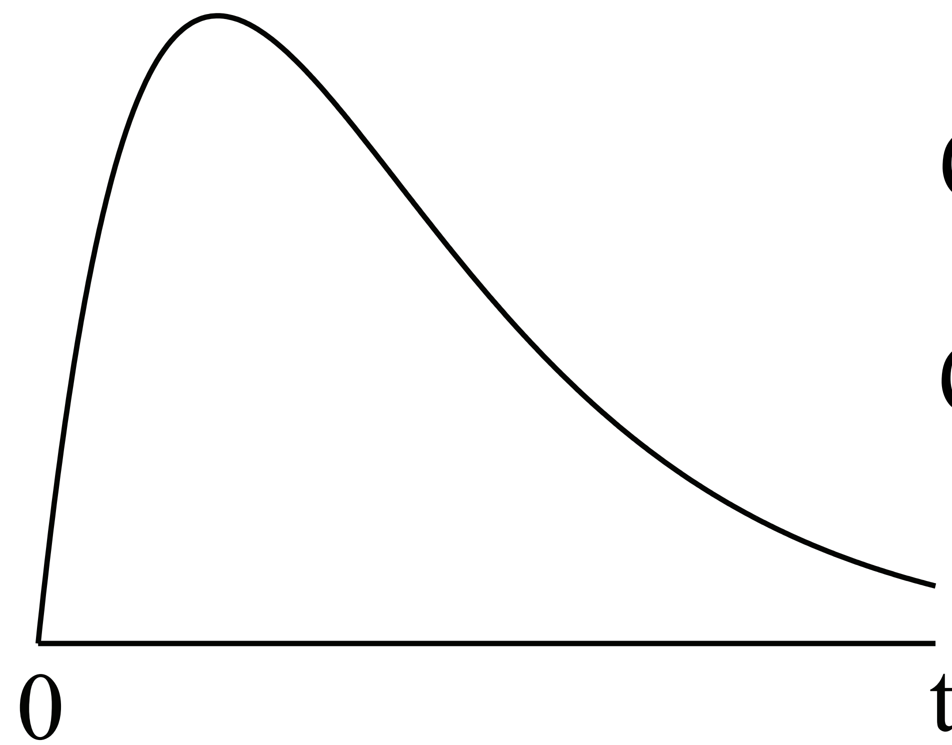


Firing rate activity $f(I)$

$$\dot{E} = -\frac{E}{\tau_E} + W_{EE} g_{EE}(A^+ - E) + W_{EI} g_{EI}(A^- - E) + P_E$$

$$\dot{I} = -\frac{I}{\tau_I} + W_{II} g_{II}(A^- - I) + W_{IE} g_{IE}(A^+ - I) + P_I$$





$$\eta(t) = \alpha^2 t e^{-\alpha t}$$

$$Q\eta = \delta$$

$$Q = \left(1 + \frac{1}{\alpha}\frac{d}{dt}\right)^2$$

$$Qg_{jE} = f(E) \qquad Qg_{jI} = f(I)$$

Steady state approximation

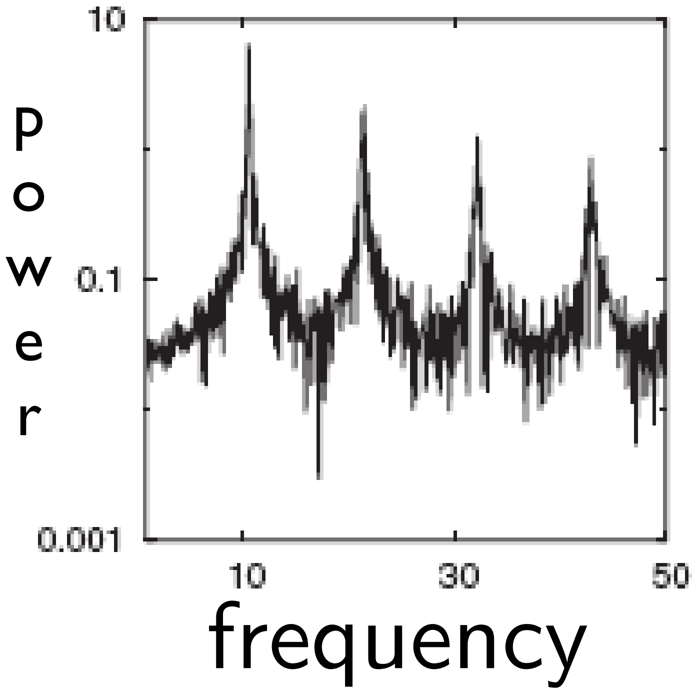
$$E = E(g_{EE}, g_{EI}) \qquad I = I(g_{II}, g_{IE})$$

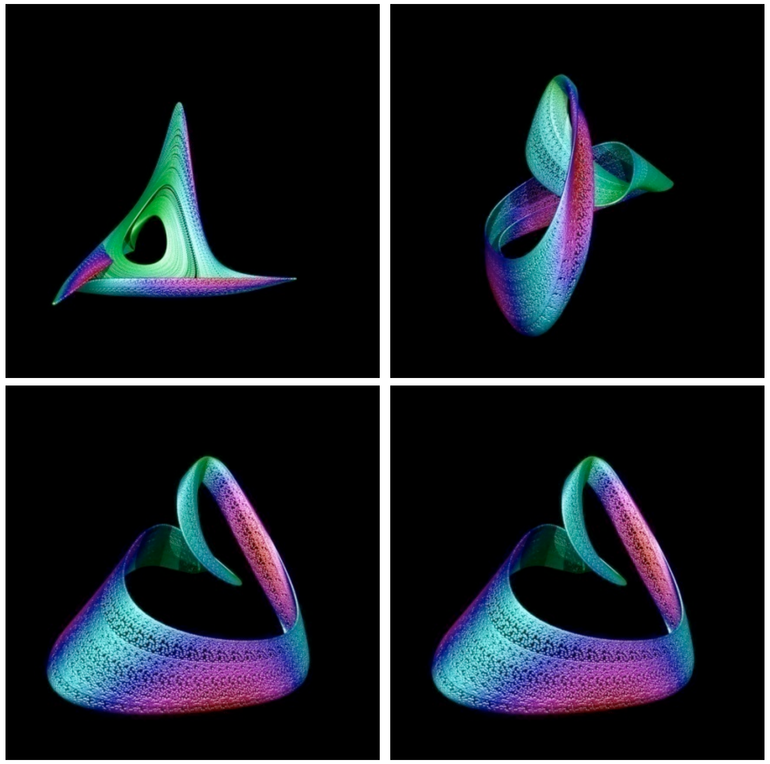
$$Qg = f$$

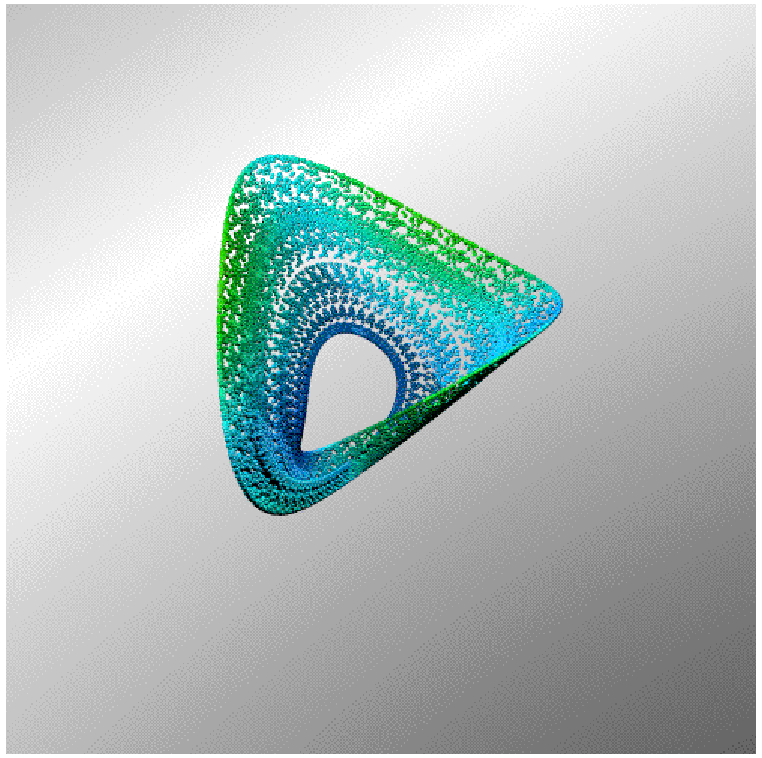
$$f = f(\{g\})$$

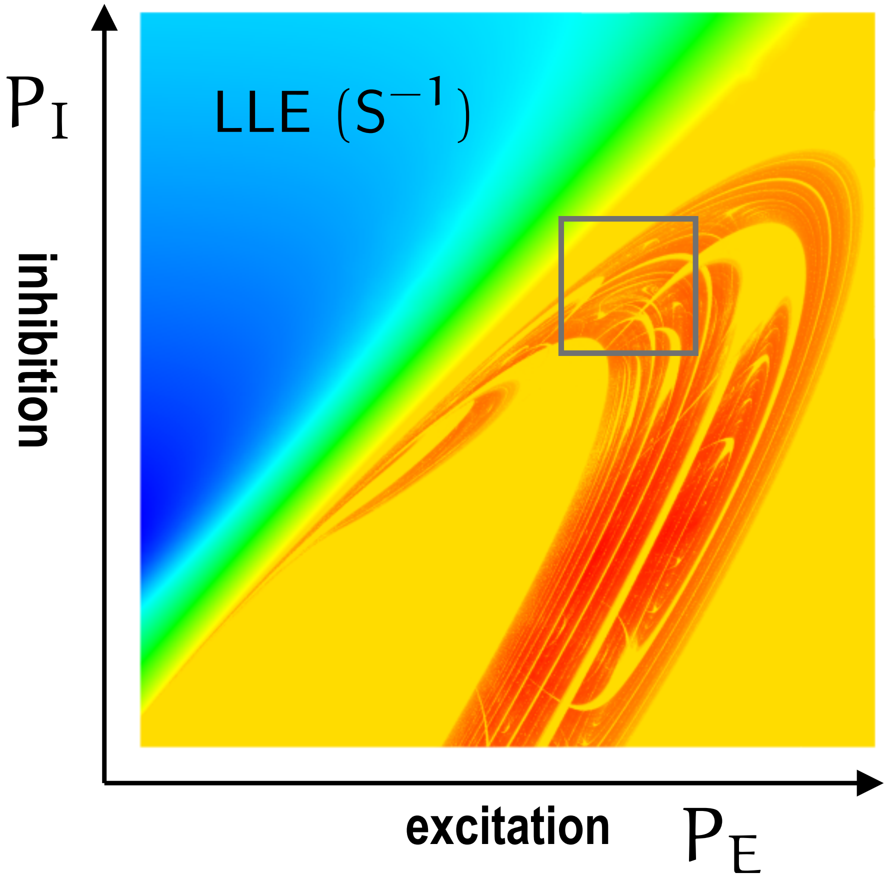
$$g = \eta * f$$

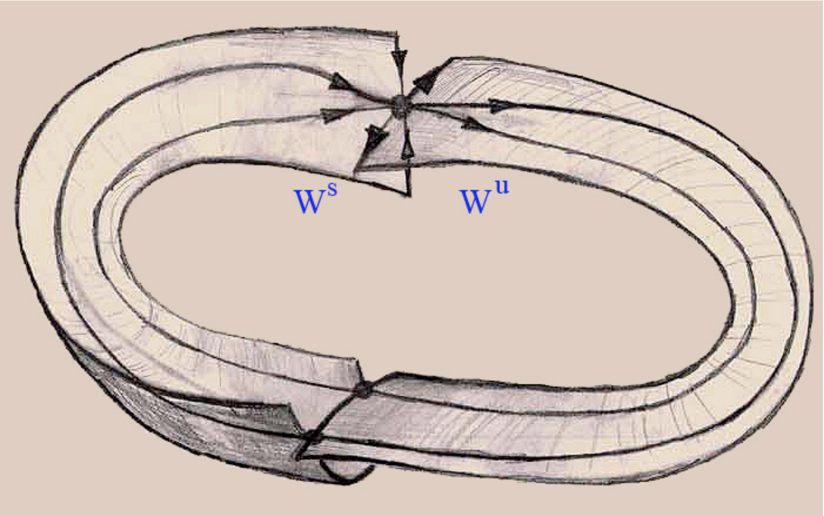
Alphoid chaos (10 D)











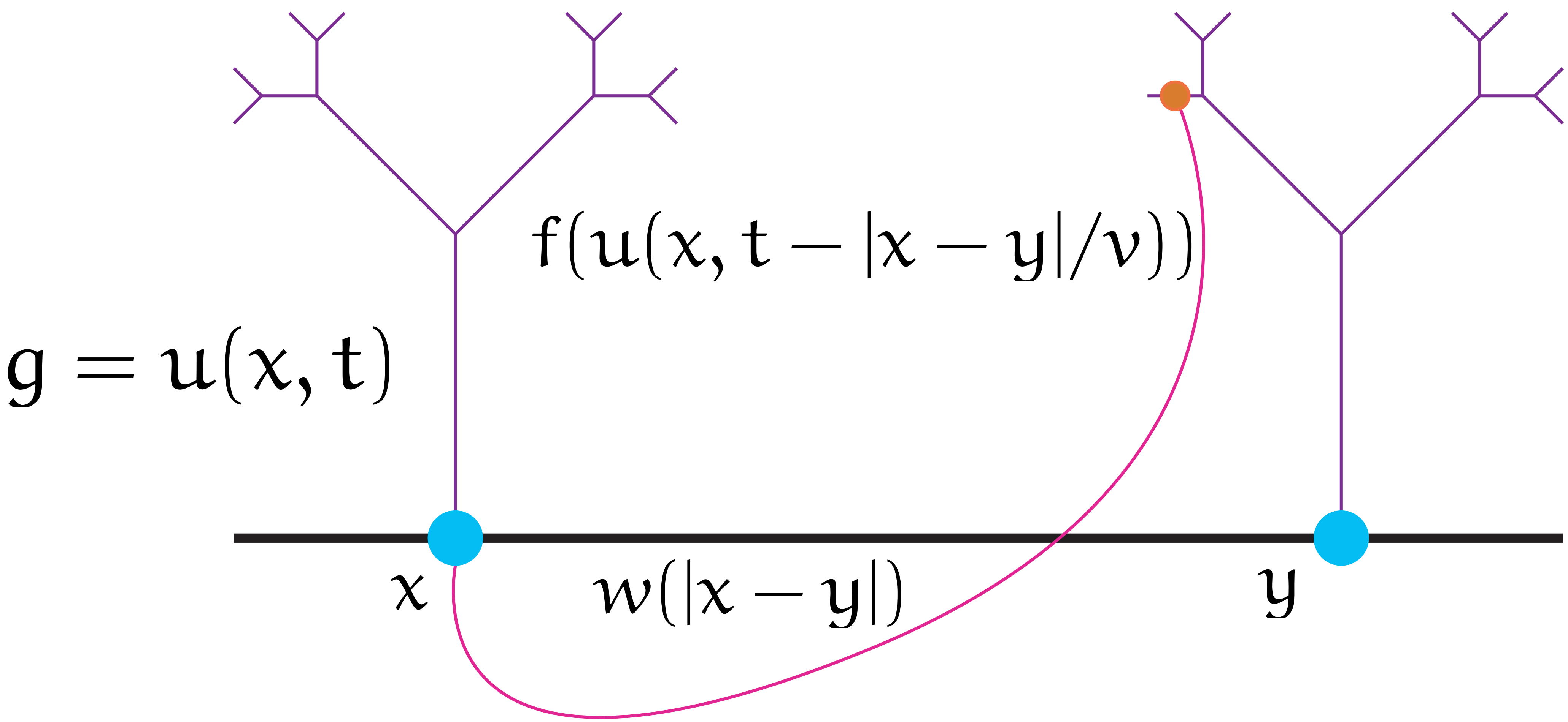
Shilnikov saddle-node route to chaos

van Veen and Liley, PRL, **97**, 208101 (2006)

Spatially extended models

$$g = w \otimes \eta * f$$

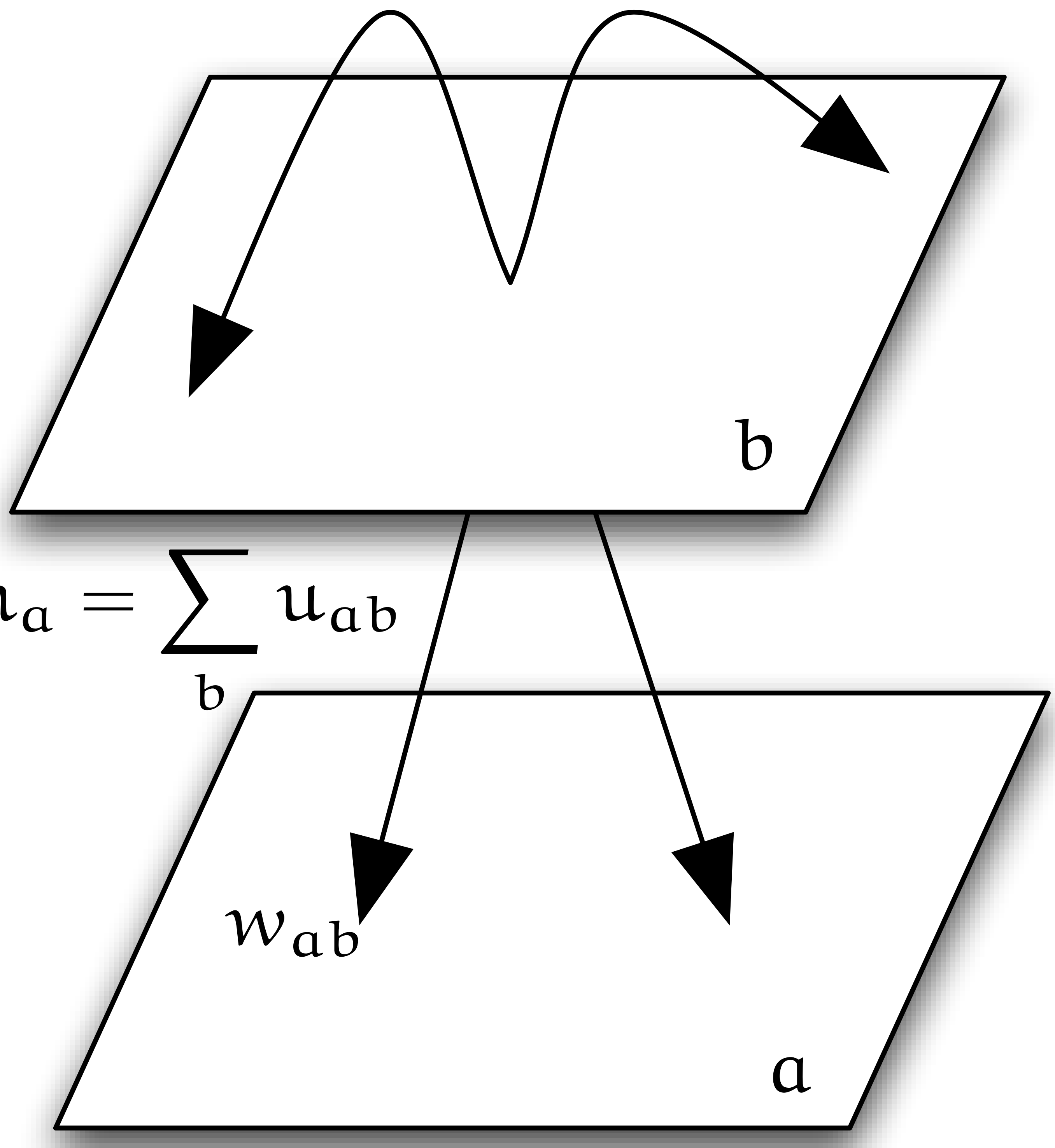
Simplest neural field model: Wilson-Cowan ('72), Amari ('77)



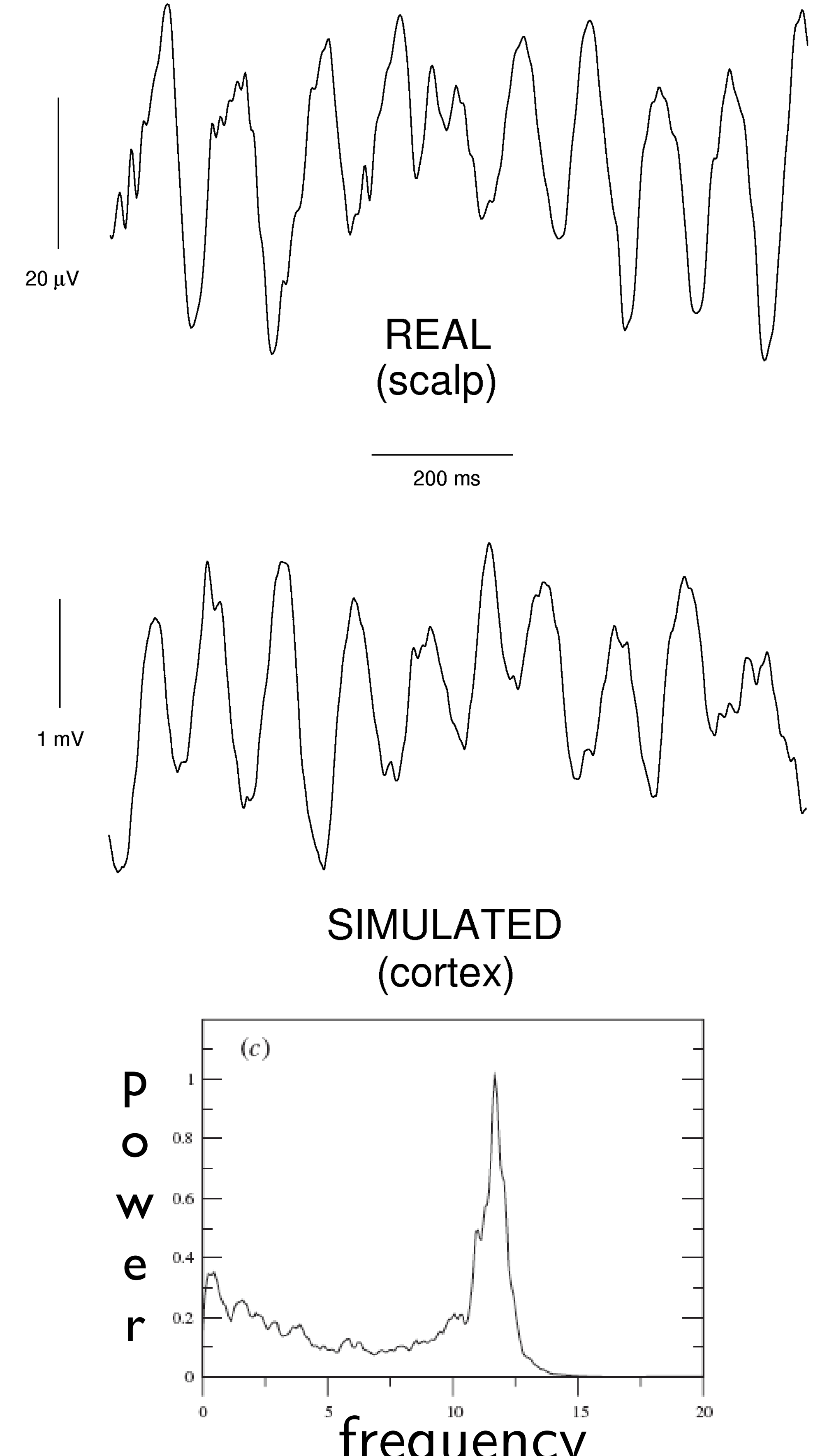
$$u(x,t) = \int_{-\infty}^{\infty} dy \; w(y) \int_{0}^{\infty} ds \; \eta(s) \; f\left(u(x-y, t-s-|y|/v)\right)$$

2D layers

$$u_{ab} = \eta_{ab} * \psi_{ab}$$



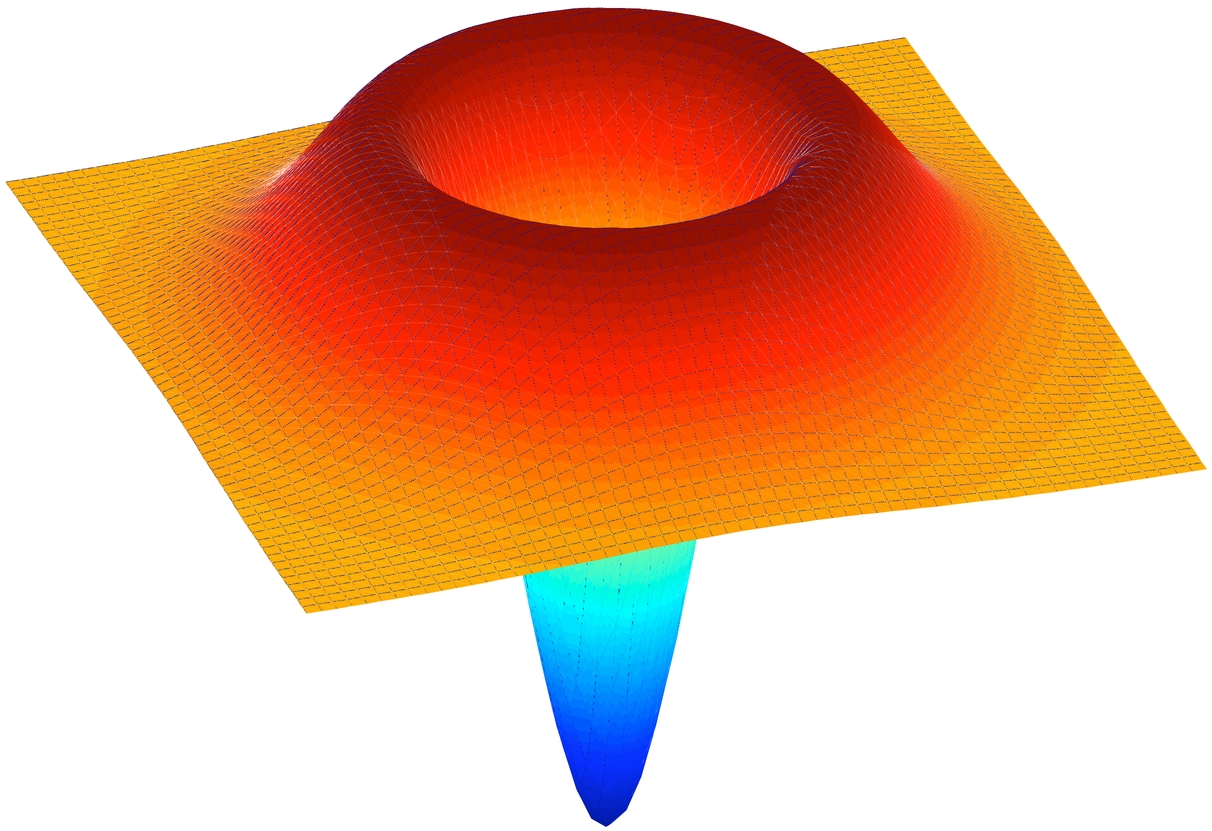
$$h_a = \sum_b u_{ab}$$



$$\psi_{ab}(\mathbf{r},t) = \int_{\mathbb{R}^2} d\mathbf{r}'\; w_{ab}(\mathbf{r},\mathbf{r}') f_b \circ h_b\left(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/\nu_{ab}\right)$$

Turing instability analysis

E layer and I layer



$$e^{i\mathbf{k}\cdot\mathbf{r}}e^{\lambda t}$$

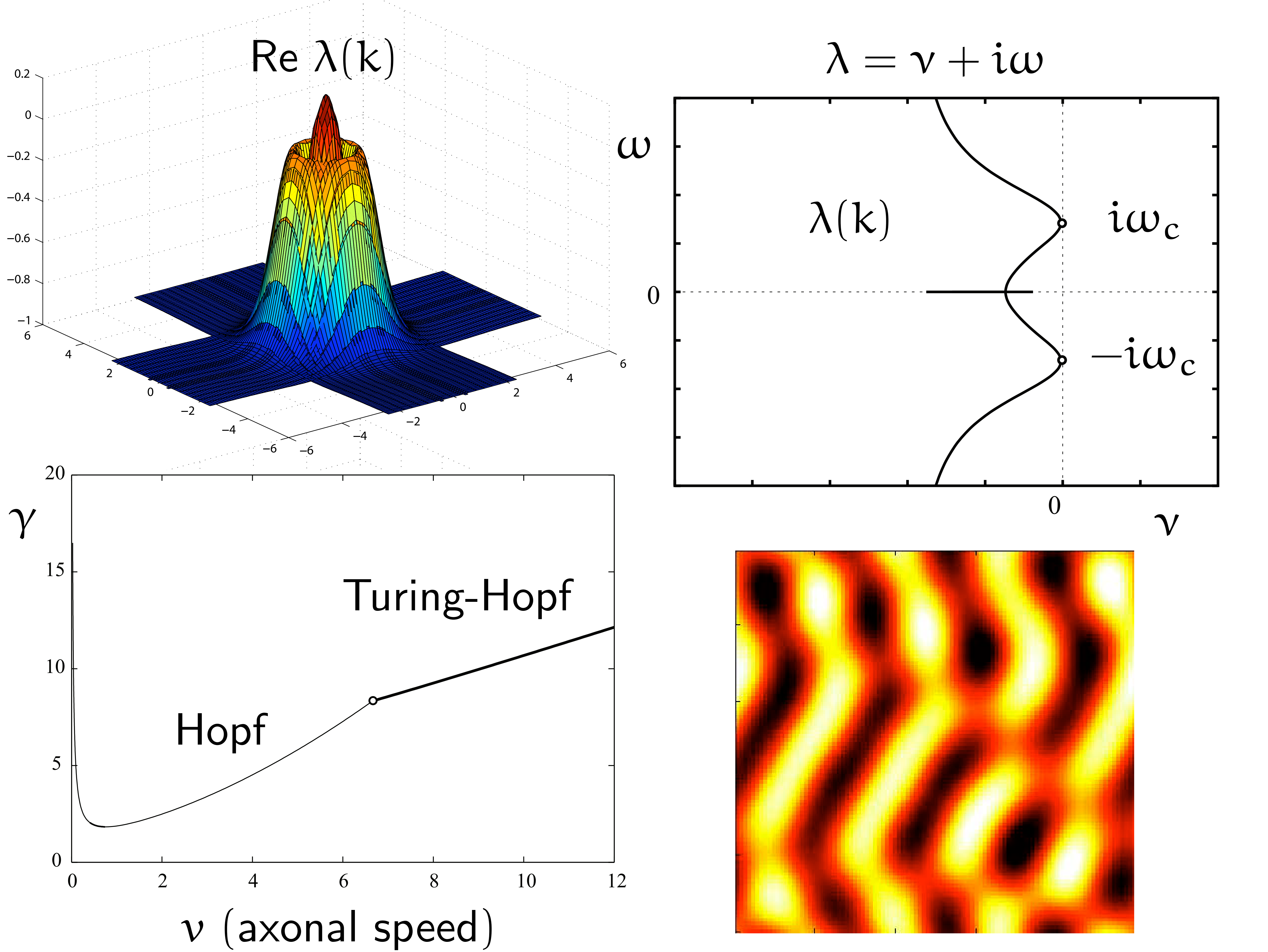
Continuous spectrum

$$\det\left(\mathcal{D}(k,\lambda) - I\right) = 0$$

$$[\mathcal{D}(k,\lambda)]_{ab} = \widetilde{\eta}_{ab}(\lambda) G_{ab}(k, -i\lambda)\gamma_b$$

$\widetilde{\eta} = \text{LT } \eta$ $\quad$ $G = \text{FLT } w(r)\delta(t - r/v)$ $\quad$ $\gamma = f'(ss)$

S Coombes et al., PRE, **76**, 05190 (2007)



Re λ(k)
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
6 4 2 0 −2 −4 −6
−6 −4 −2 0 2 4 6
λ = ν + iω
ω
λ(k)
iω_c
−iω_c
0
0
ν
γ
20
15
10
5
0
Turing-Hopf
Hopf
0 2 4 6 8 10 12
ν (axonal speed)

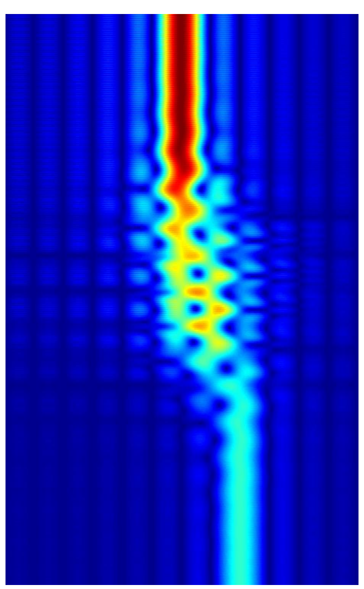
Amplitude Equations (one D)

Coupled mean-field Ginzburg–Landau equations describing a Turing–Hopf bifurcation with modulation group velocity of $O(1)$.

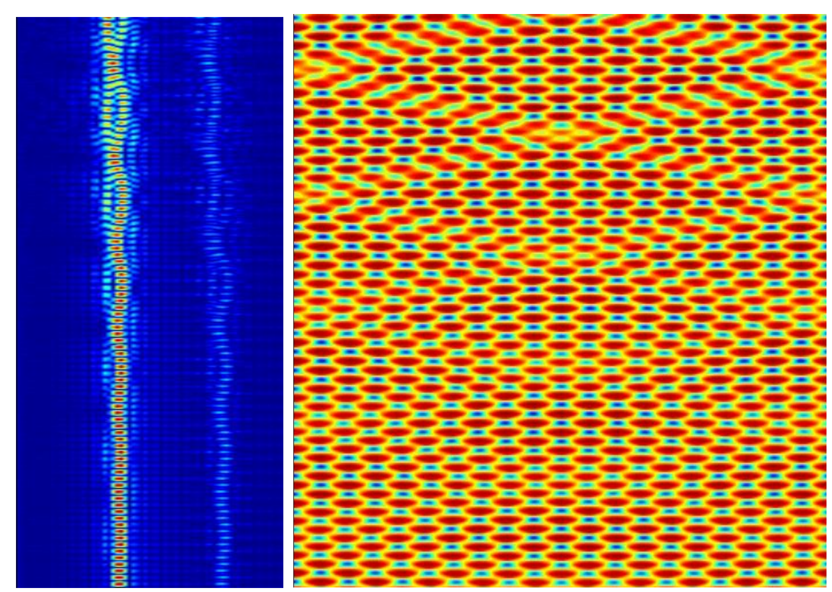
$$\frac{\partial A_1}{\partial \tau} = A_1(a + b|A_1|^2 + c\langle |A_2|^2 \rangle) + d\frac{\partial^2 A_1}{\partial \xi_+^2}$$

$$\frac{\partial A_2}{\partial \tau} = A_2(a + b|A_2|^2 + c\langle |A_1|^2 \rangle) + d\frac{\partial^2 A_2}{\partial \xi_-^2}$$

Benjamin–Feir (BF)

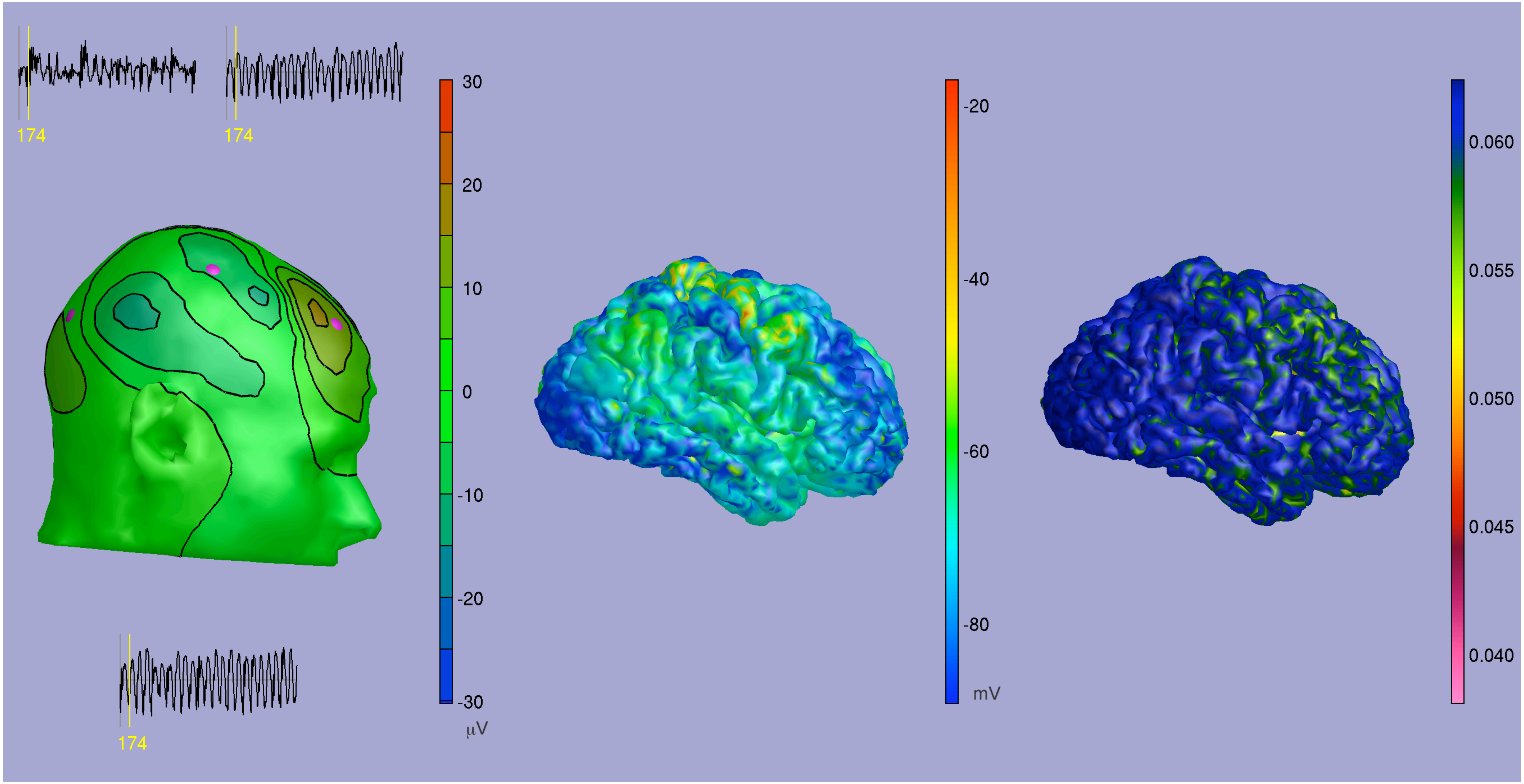


BF-Eckhaus instability



Coefficients in terms of integral transforms of w and η .

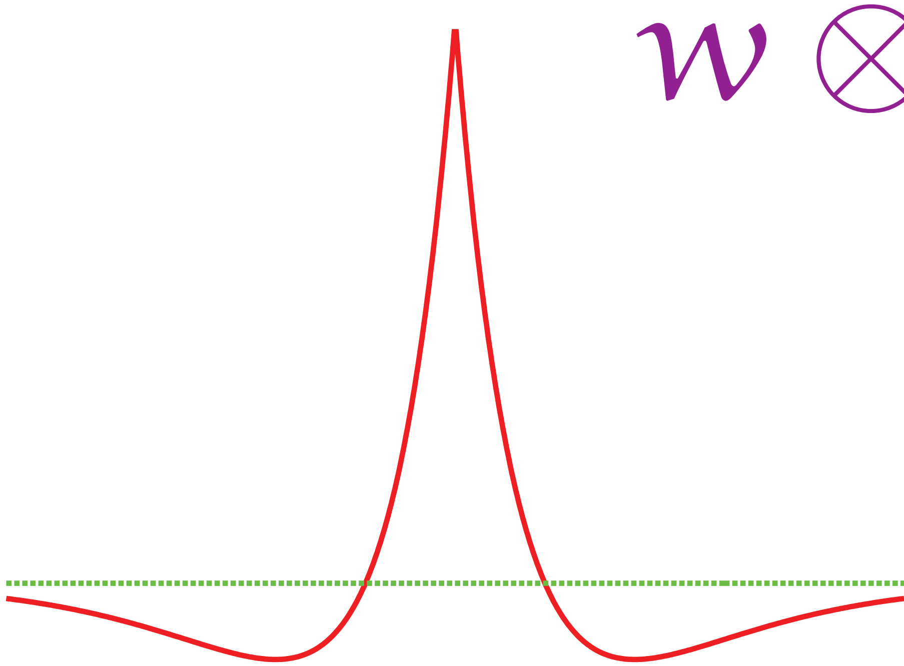
Applications to co-registered EEG/fMRI



Bojak, I., Oostendorp, T. F., Reid, A. T., Kotter, R., 2009. Realistic mean field forward predictions for the integration of co-registered EEG/fMRI. BMC Neuroscience 10, L2.

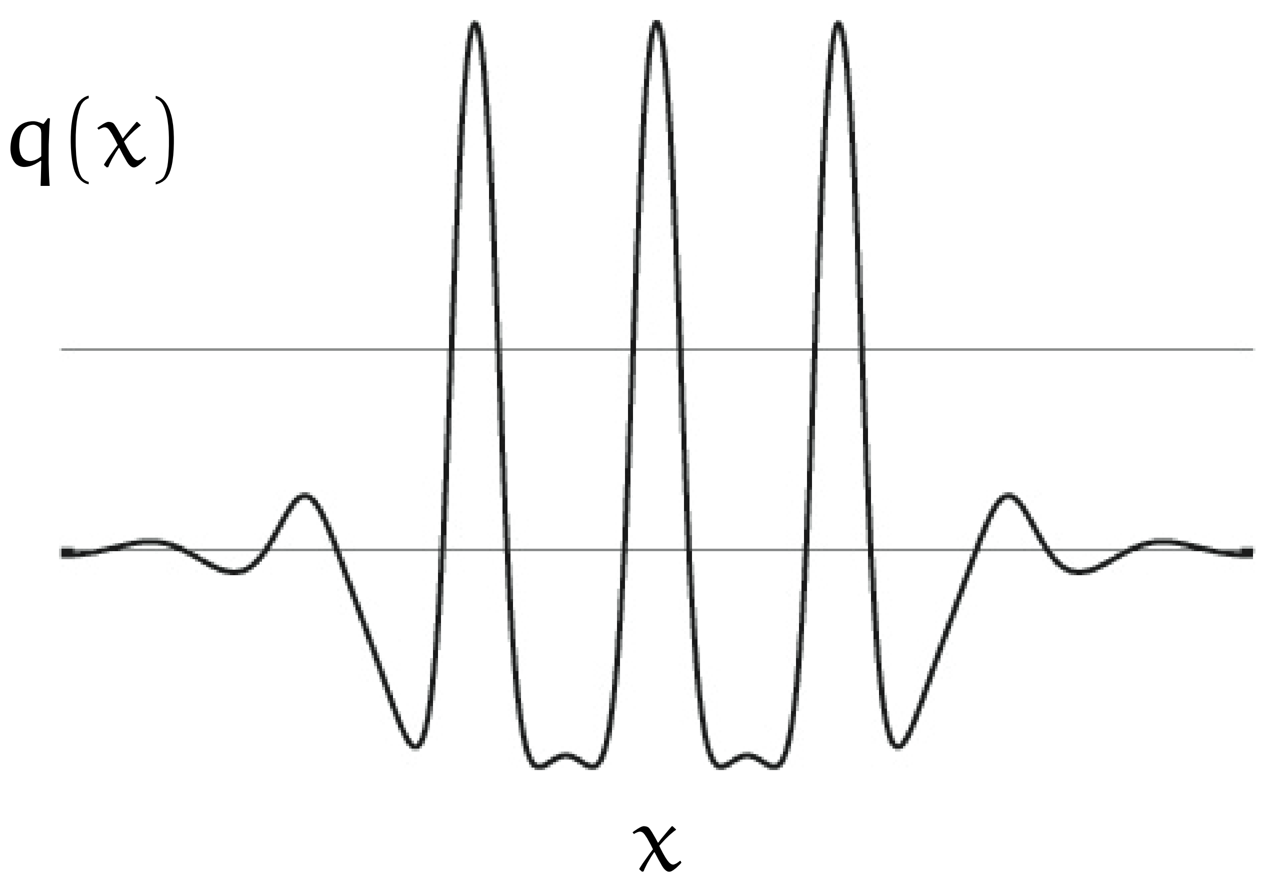
Time independent localised solutions

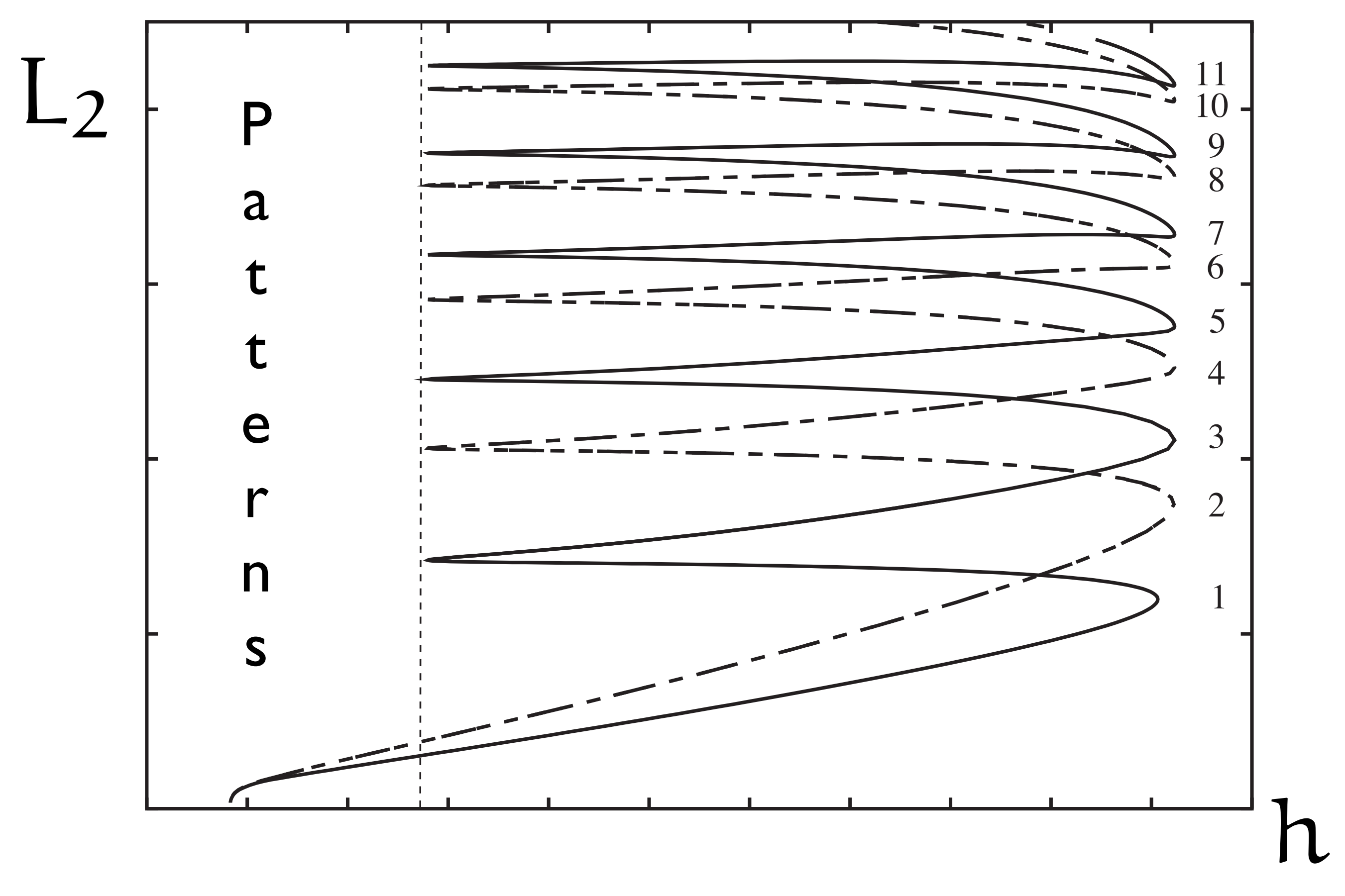
$$w \otimes \eta * f \rightarrow w \otimes f$$



$$q(x) = \int_{\mathbb{R}} dy \; w(x-y) f \circ q(y)$$

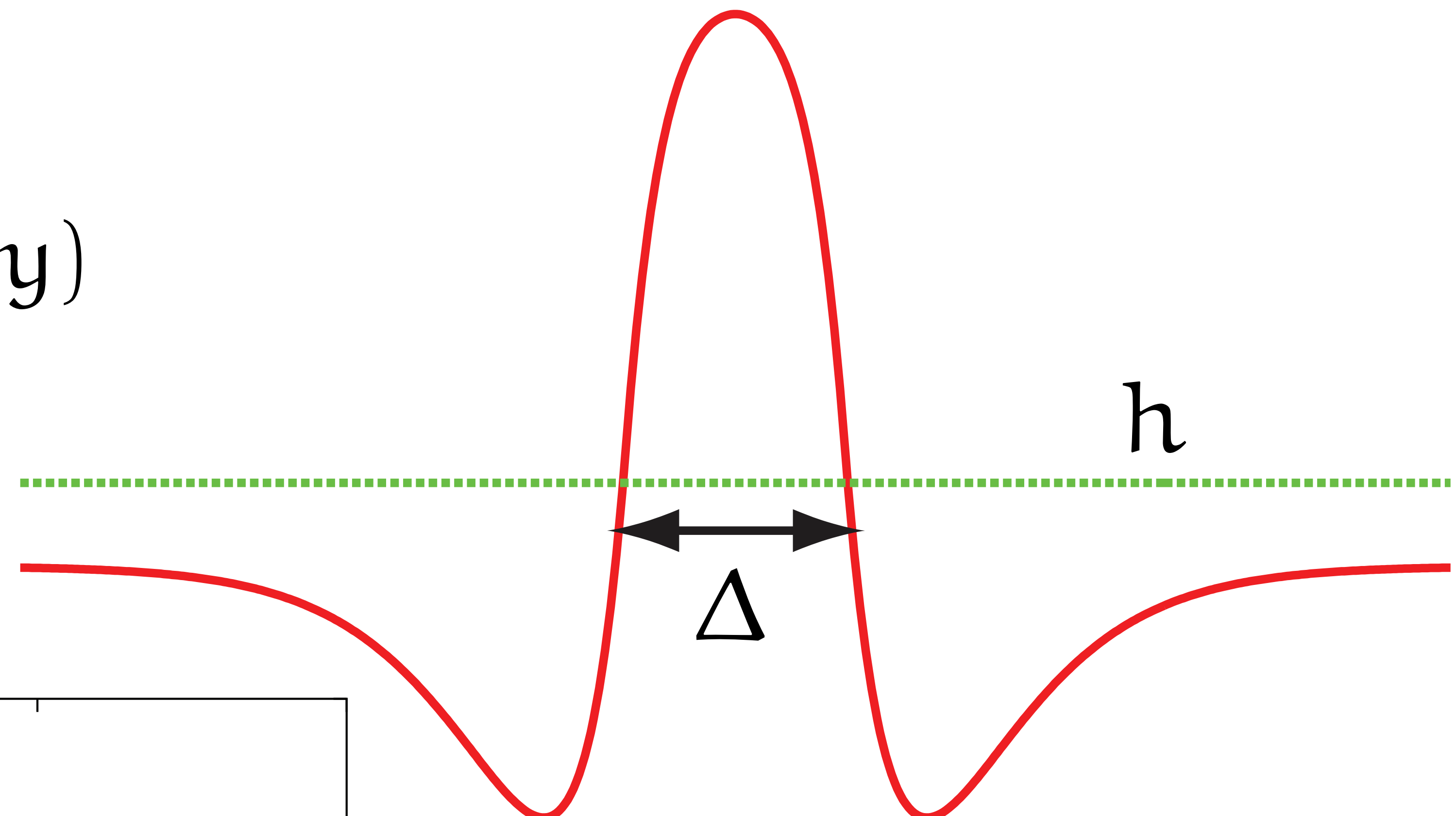
$$w(x) = (1 - |x|)e^{-|x|}$$

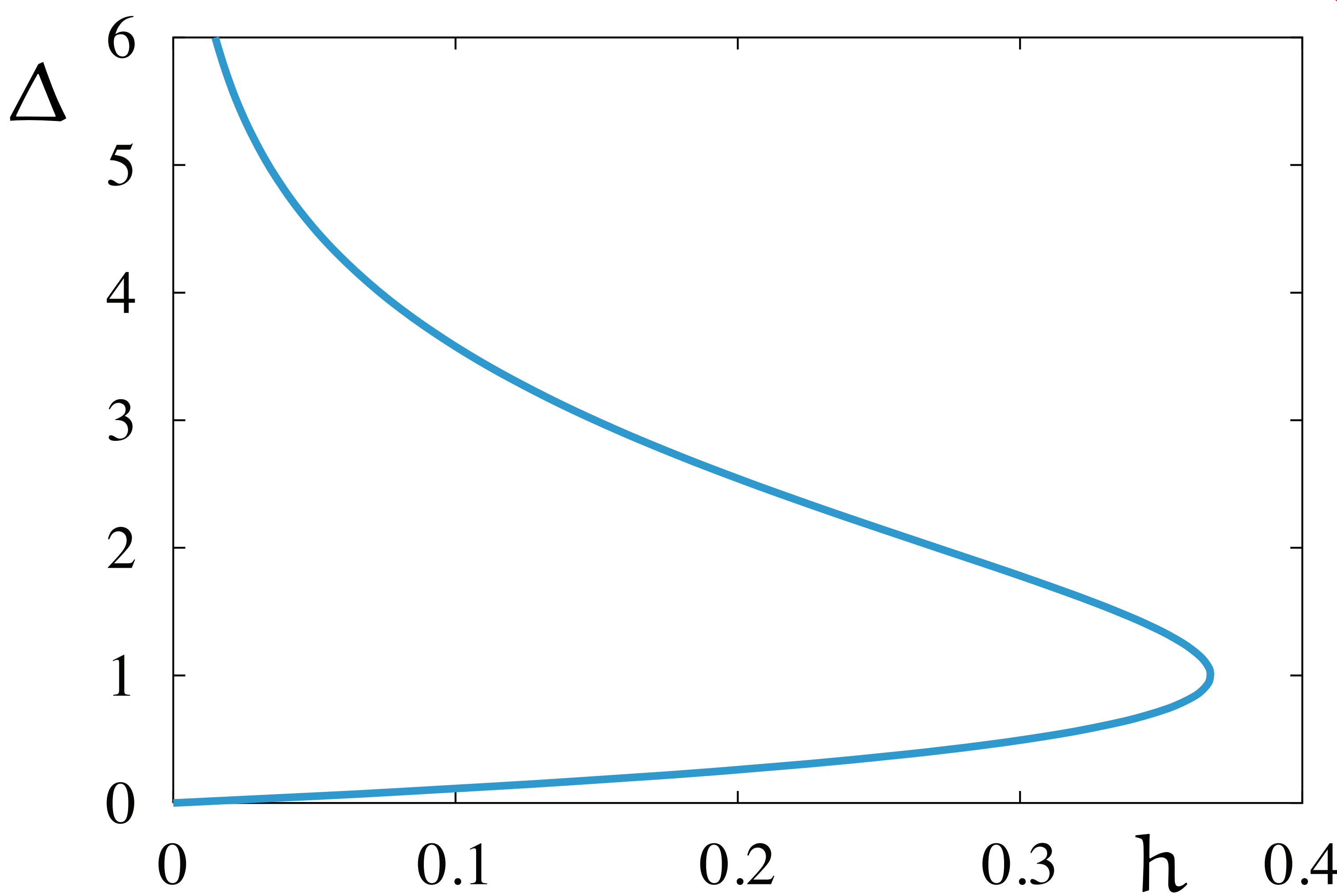




Exact result for 1-bump: $f(u) = H(u - h)$

$$q(x) = \int_0^{\Delta} dy\ w(x - y)$$





$$q(0) = h = q(\Delta)$$

$$\Delta e^{-\Delta} = h$$

working memory

Stability

Examine eigenspectrum of the linearization about a solu

Solutions of form $u(x)e^{\lambda t}$ satisfy $\mathcal{L}u(x) = u(x)$

$$\mathcal{L}u(x) = \tilde{\eta}(\lambda) \int_{-\infty}^{\infty} dy \; w(x-y) f'(q(y)-h) u(y)$$

For Heaviside firing rate

$$f'(q(x)) = \frac{\delta(x)}{|q'(0)|} + \frac{\delta(x-\Delta)}{|q'(\Delta)|}$$

so

$$u(x) = \frac{\tilde{\eta}(\lambda)}{|w(0) - w(\Delta)|} [w(x)u(0) + w(x-\Delta)u(\Delta)]$$

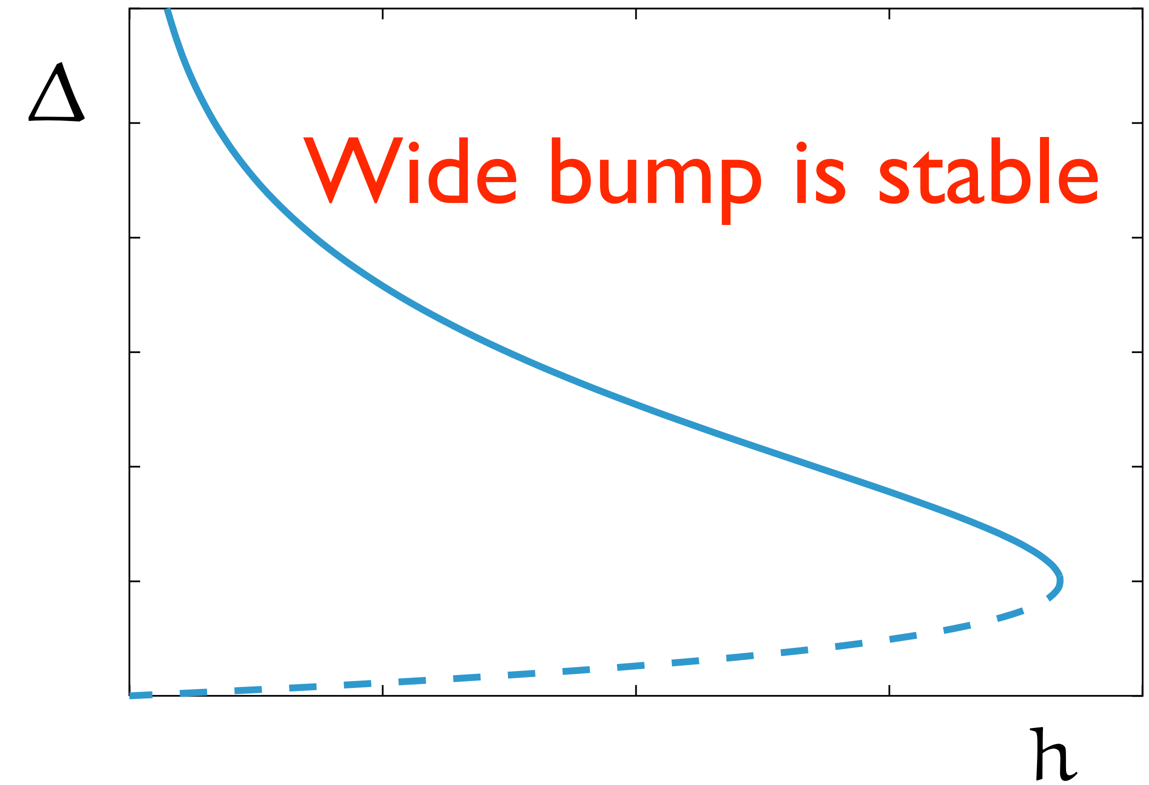
System of linear equations for perturbations at threshold

$$\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix} = \mathcal{A}(\lambda) \begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix}, \qquad \mathcal{A}(\lambda) = \frac{\widetilde{\eta}(\lambda)}{|w(0) - w(\Delta)|} \begin{bmatrix} w(0) & w(\Delta) \\ w(\Delta) & w(0) \end{bmatrix}$$

Non trivial solution if

$$\mathcal{E}(\lambda) = \det(\mathcal{A}(\lambda) - I) = 0$$

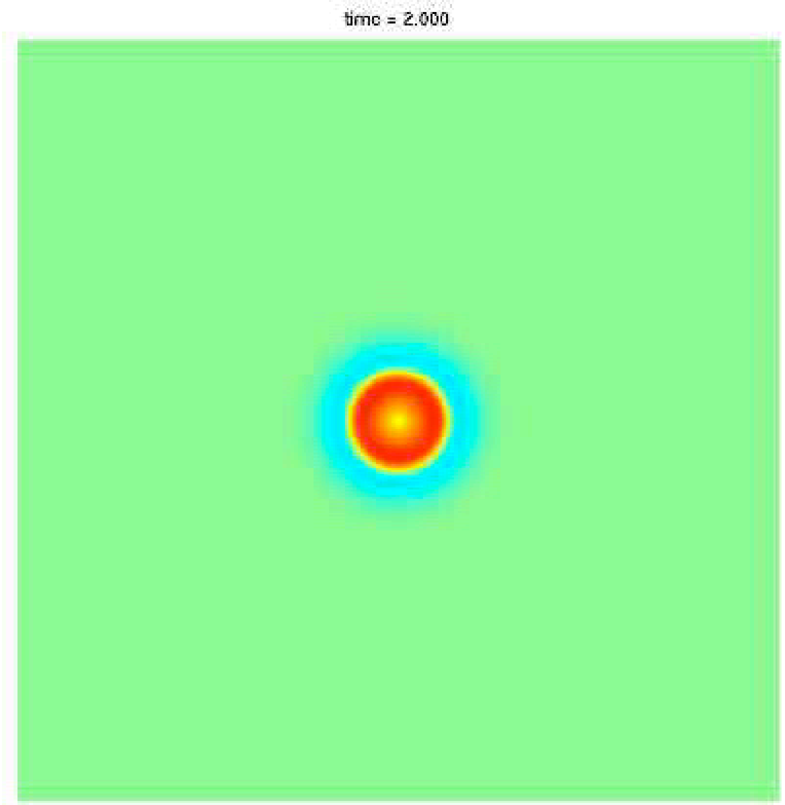
Solutions stable if Re $\lambda < 0$

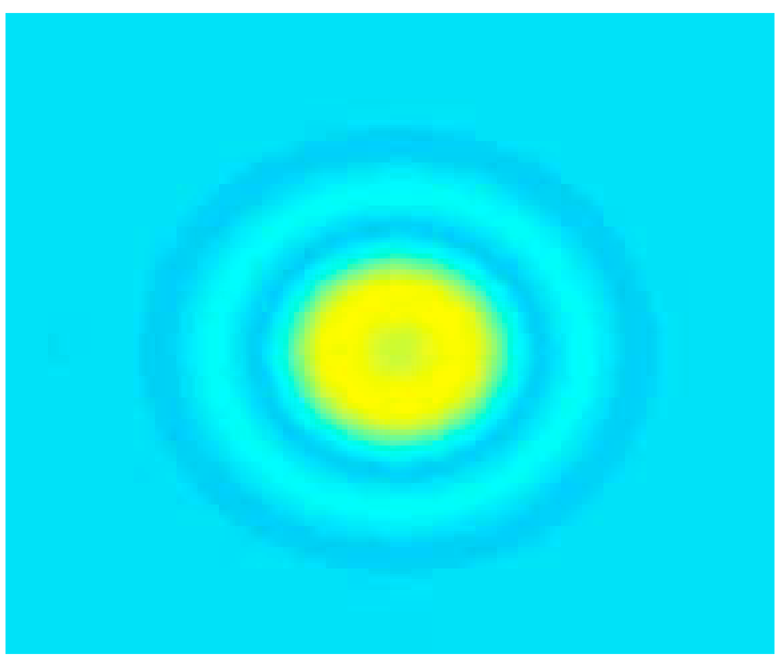


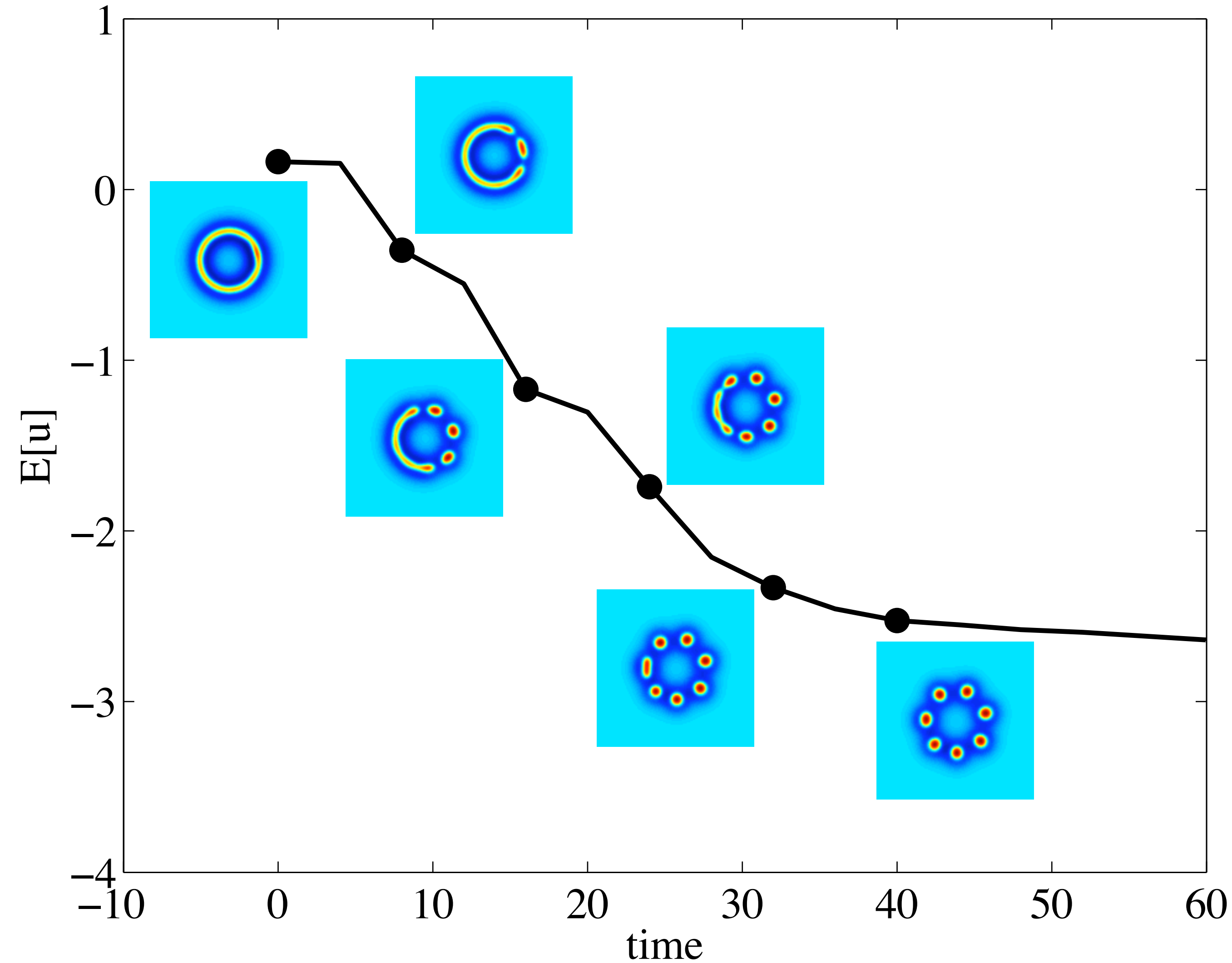
Evans function for integral neural field equation

S Coombes and M R Owen (2004) Evans functions for integral neural field equations with Heaviside firing rate function, SIAM Journal on Applied Dynamical Systems, Vol 34, 574-600.

Predictions of Evans function







M R Owen, C R Laing and S Coombes 2007 Bumps and rings in a two-dimensional neural field: splitting and rotational instabilities, New Journal of Physics, Vol 9, 378

Threshold accommodation

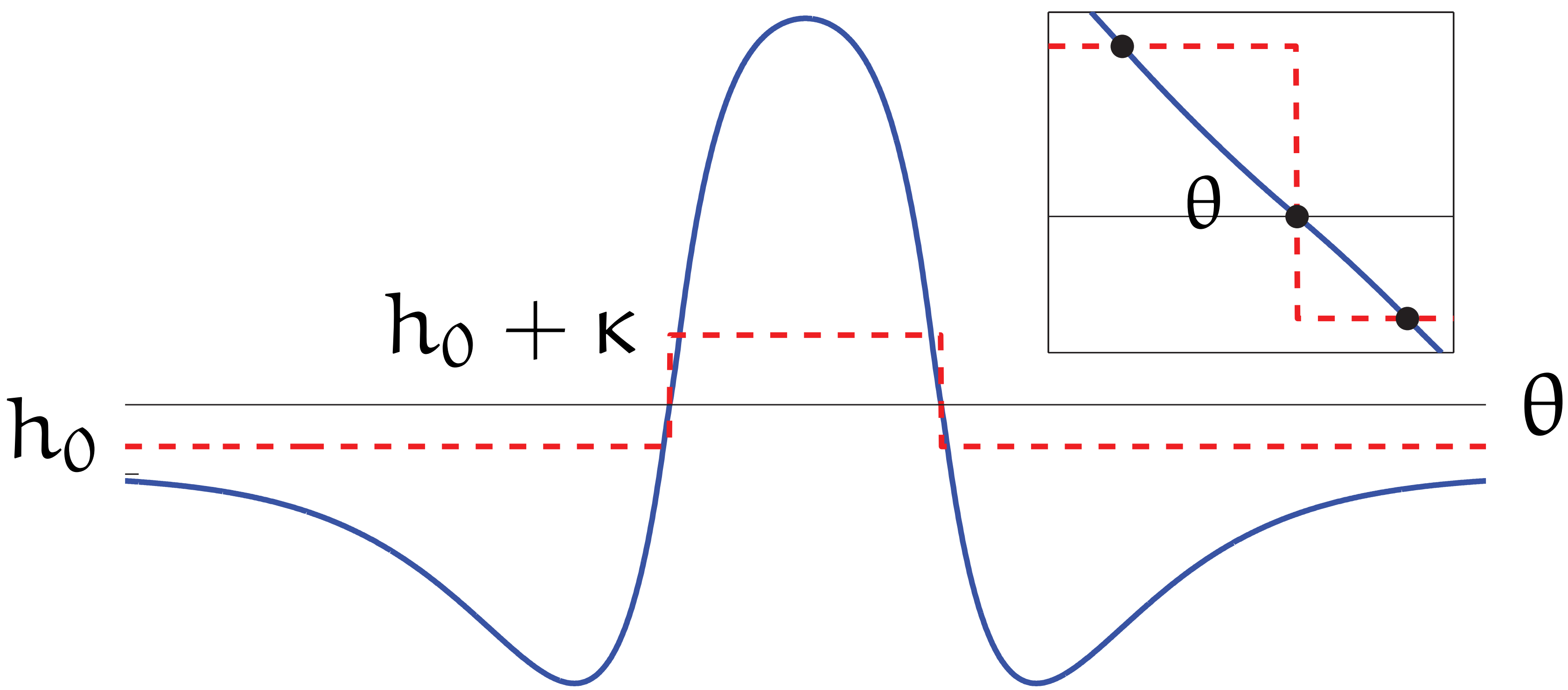
Hill (1936), “... the threshold rises when the *local potential* is maintained ... and reverts gradually to its original value when the nerve is allowed to rest.”

$$\frac{\partial h}{\partial t} = -(h - h_0) + \kappa H(u - \theta)$$

One bump $(u, h) = (q(x), p(x))$

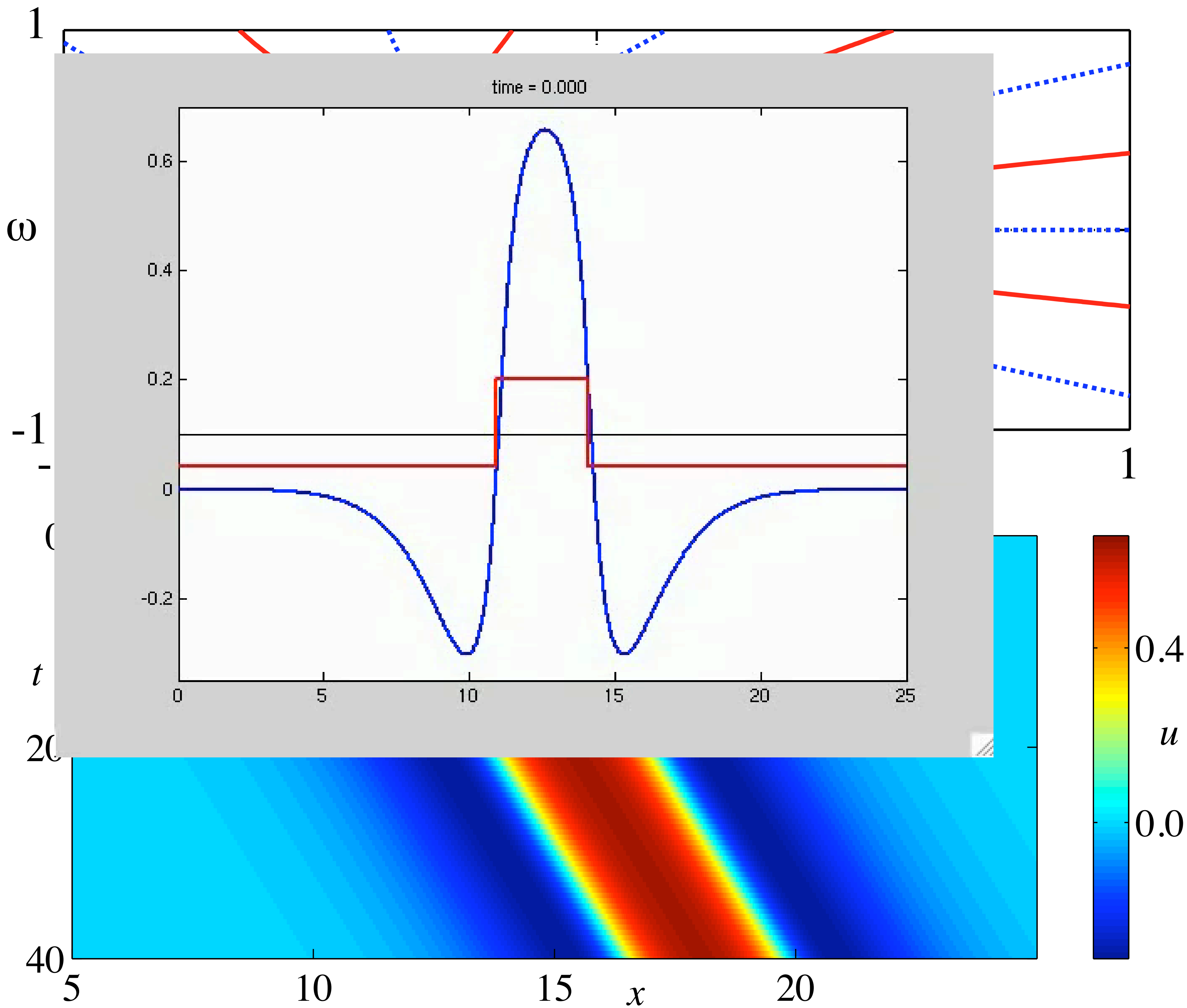
$$q = w \otimes H(q - p)$$

$$p = \begin{cases} h_0 & q < \theta \\ h_0 + \kappa & q \geq \theta \end{cases}$$



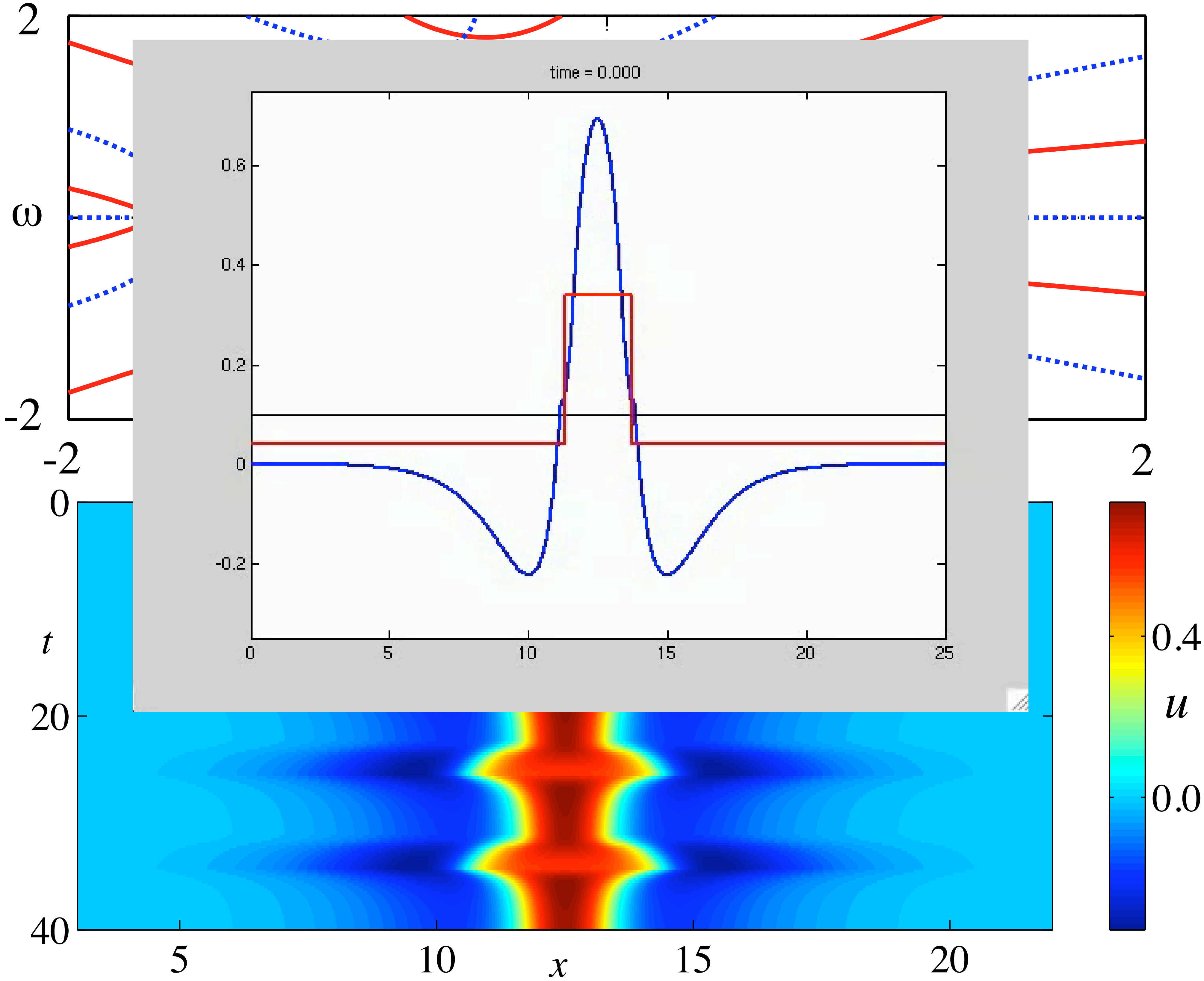
Bump Stability I: $\eta(t) = \alpha^2 t e^{-\alpha t}$

Low κ instability on Re axis (increasing α)

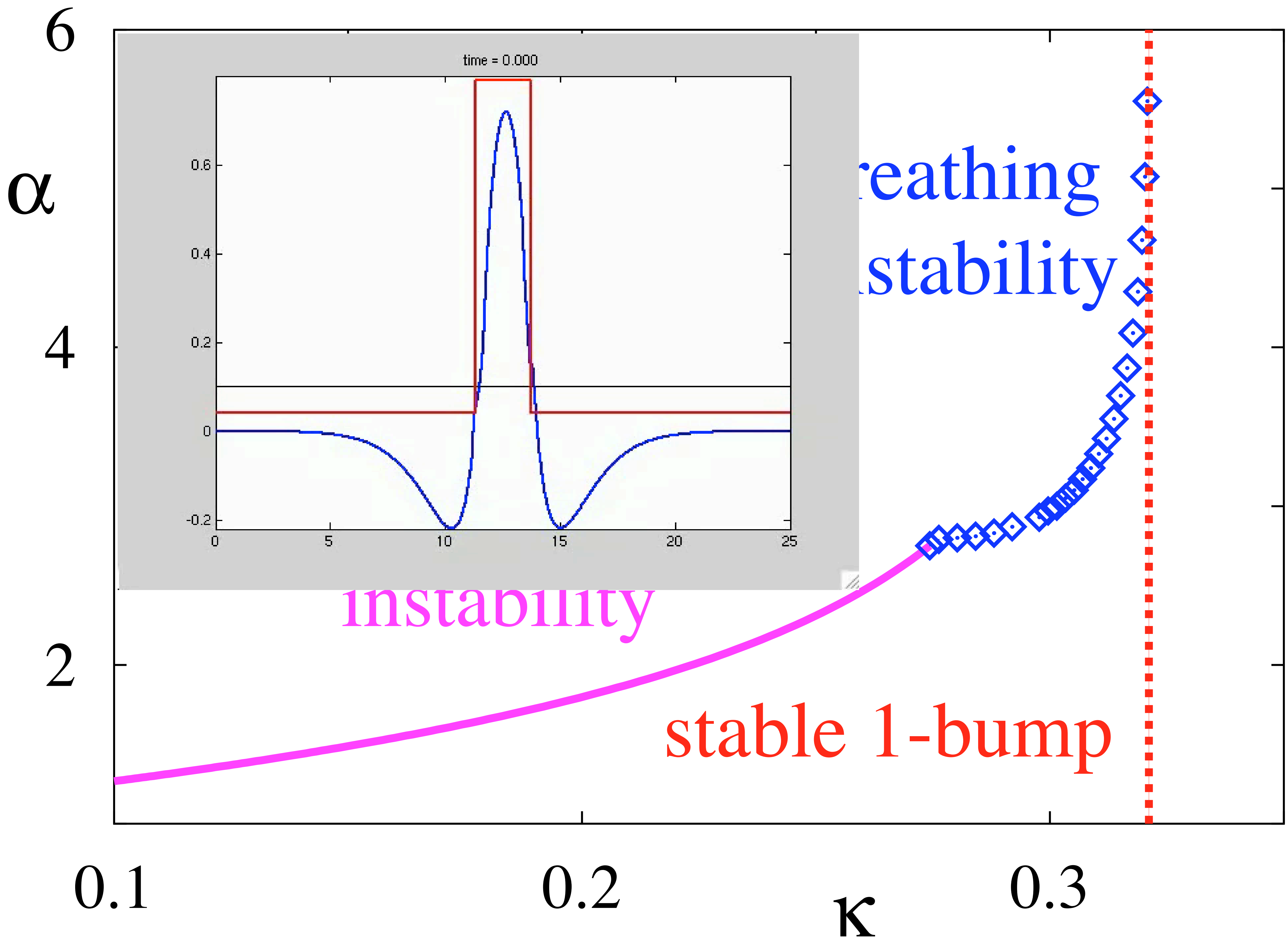


Bump Stability II

High κ instability on Im axis (increasing α) gives a breather

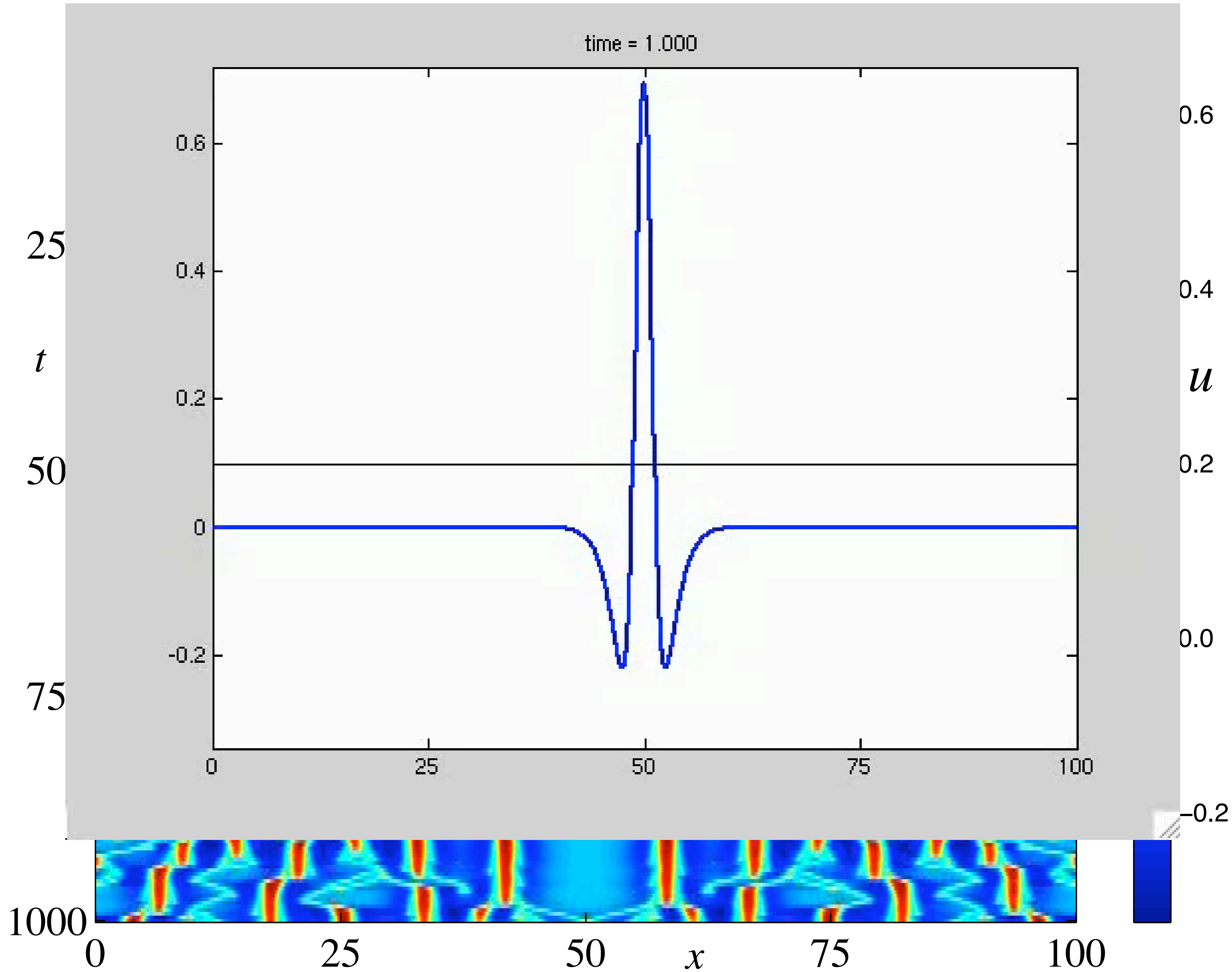


Summary of Bump instabilities



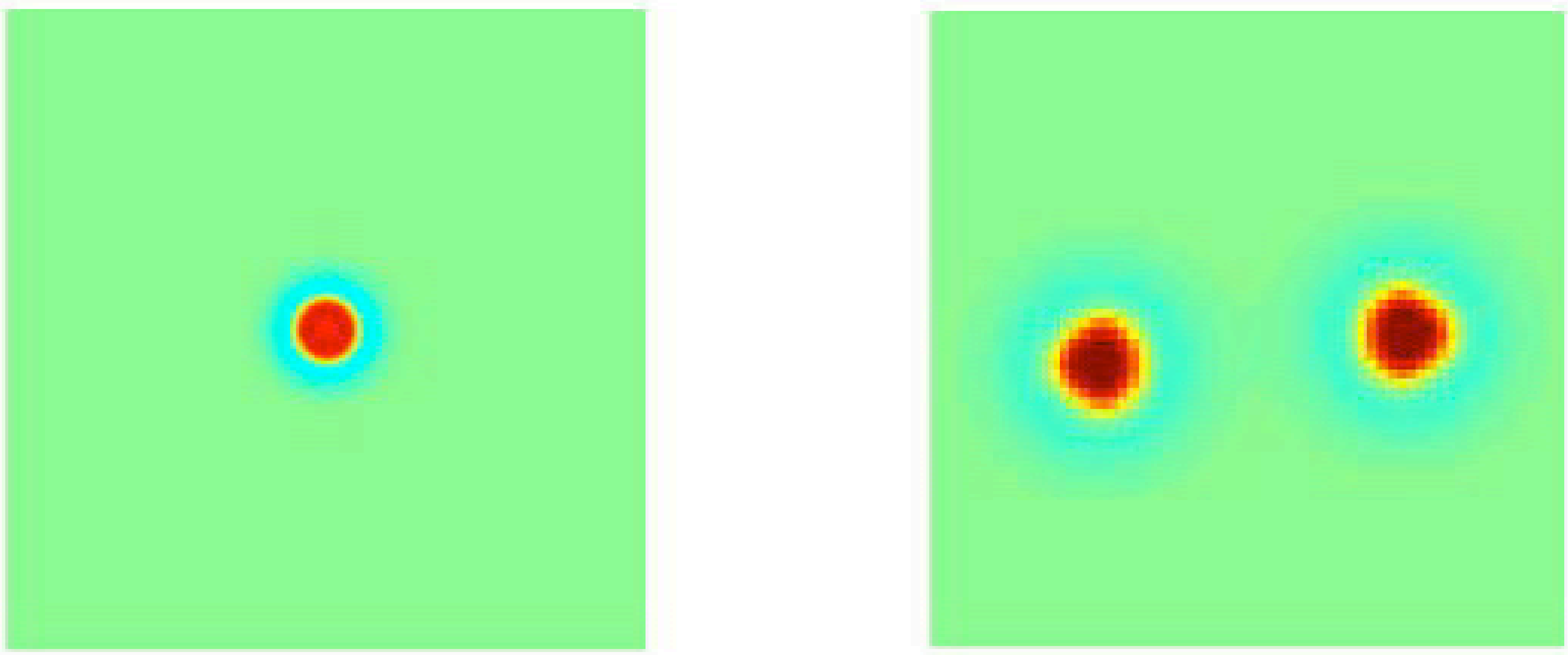
Exotic Dynamics

... including asymmetric breathers, multiple bumps, multiple pulses, periodic traveling waves, and bump-splitting instabilities that appear to lead to spatio-temporal chaos.



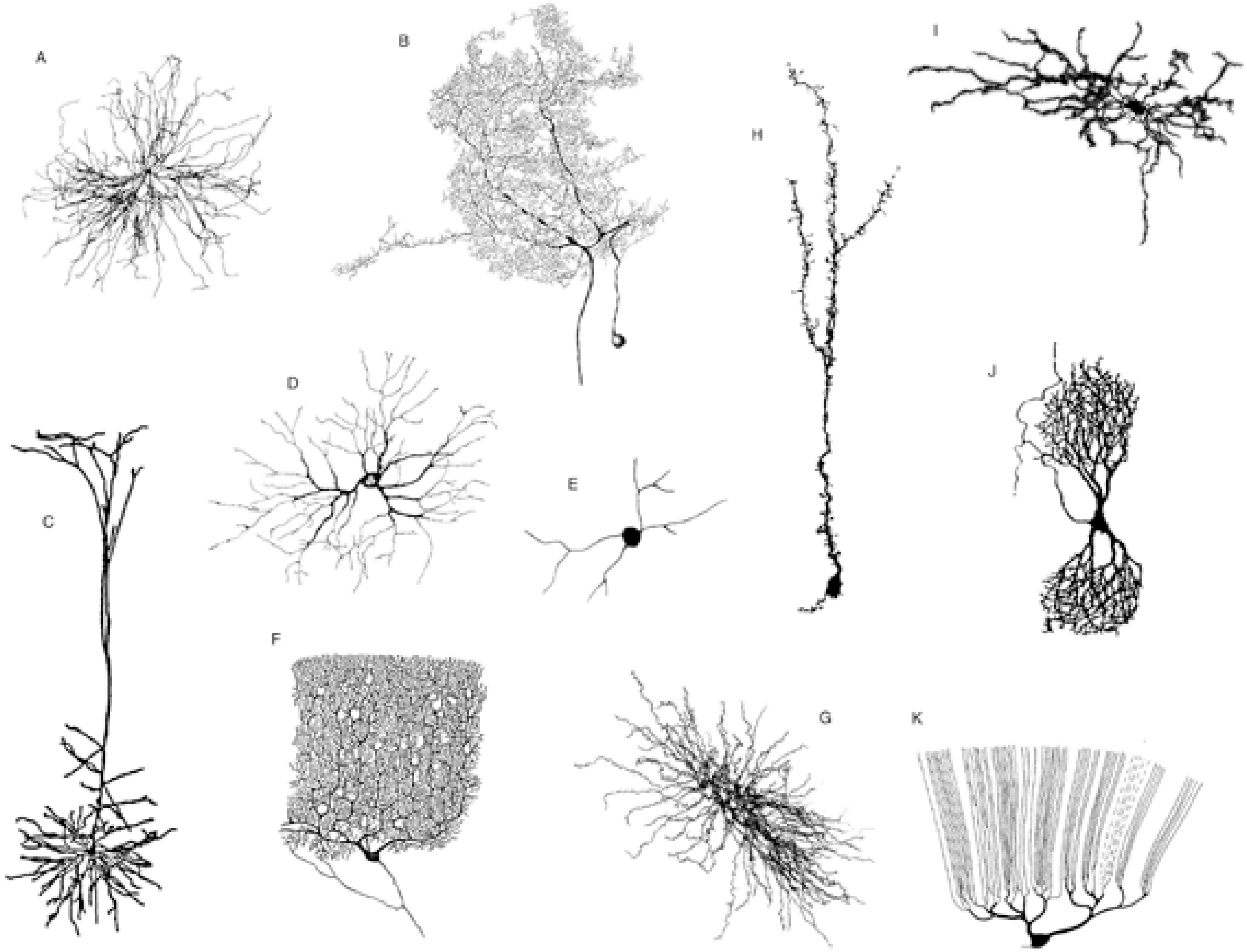
S Coombes and M R Owen: Bumps, breathers and waves in a neural network with spike frequency adaptation. PRL, 94, 148102, (2005).

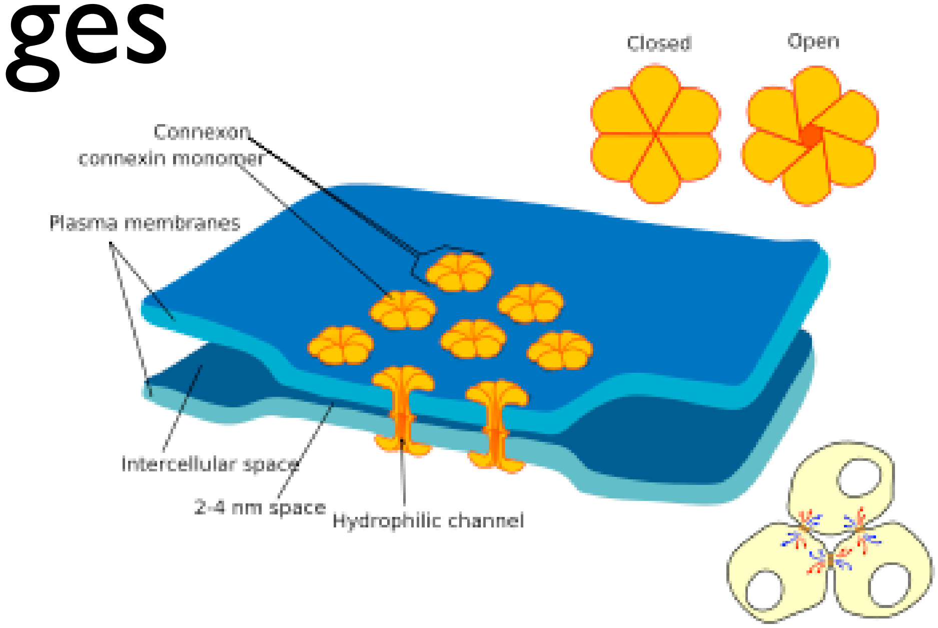
Splitting and scattering

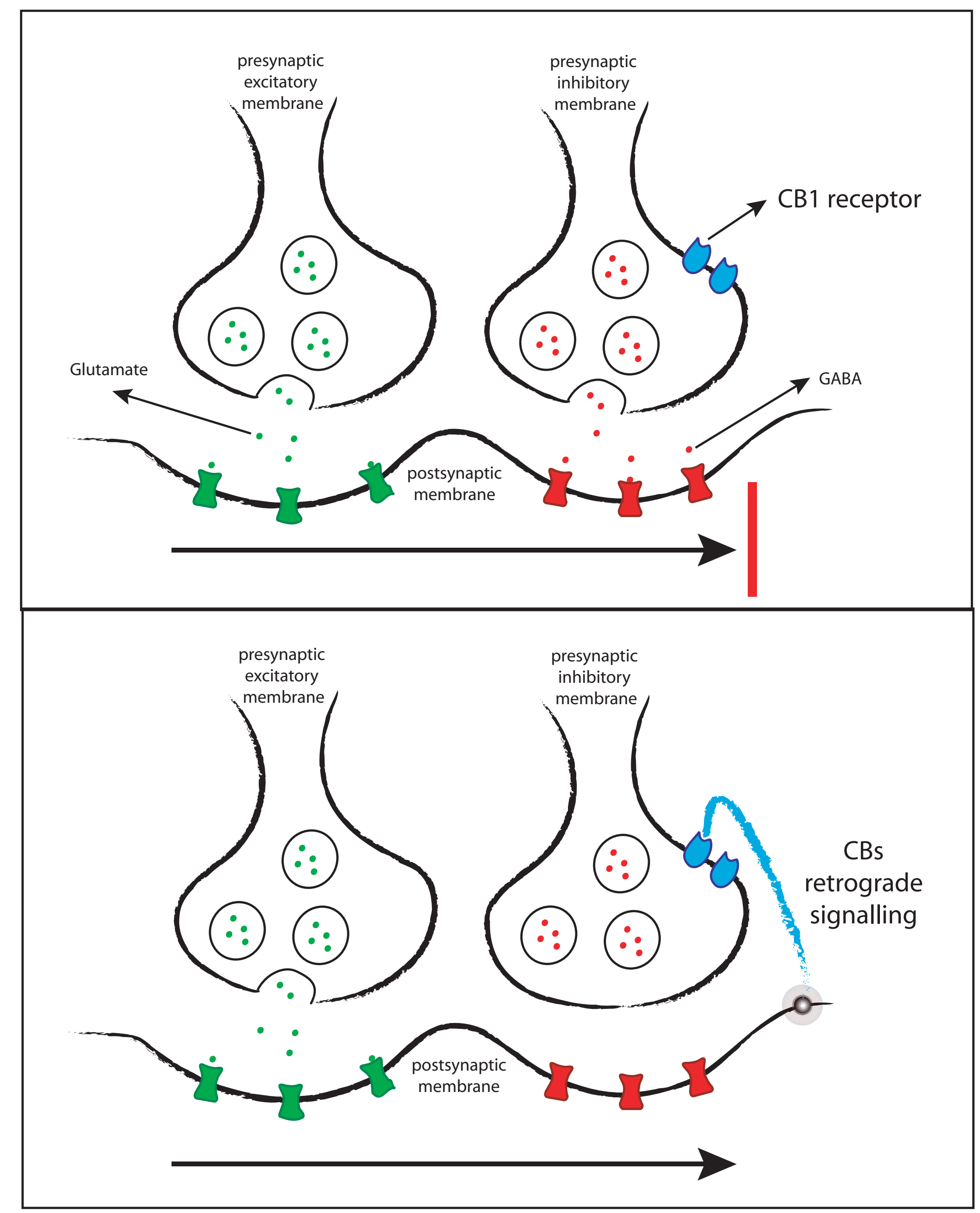


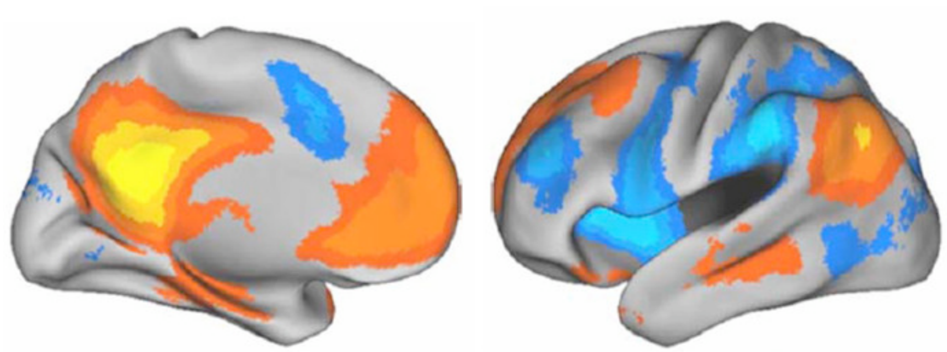
Auto/dispersive solitons as seen in coupled cubic complex Ginzburg-Landau systems and three component reaction-diffusion systems.

Further Challenges









Default mode network and ultra slow coherent oscillations

In collaboration with

Nikola Venkov
(Notts)

Gabriel Lord
(Heriot-Watt)

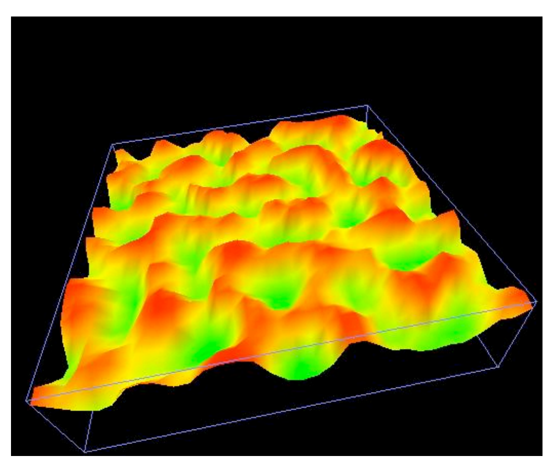
Yulia Timofeeva
(Warwick)

David Liley
(Melbourne)

EPSRC Engineering and Physical Sciences Research Council

Markus Owen (Notts)

Ingo Bojak (Nijmegen)



Carlo Laing (Massey, NZ)