Hyperbolic systems
in
coupled map networks

E. Fontich (UB), P. Martin (UPC)

R. de la Llave

The University of Texas at Austin

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Related work with M. Jiang, Y. Sire

We consider coupled map lattices, which is a problem that has been widely studied since the 80's.

$$\mathcal{M} = M^{\mathbb{Z}^d}\ , \quad M \text{ a compact manifold}$$

$$\Phi : \mathcal{M} \to \mathcal{M}\ , \quad \Phi_i(x) = f(x) \text{ uncoupled map}$$

$$F : \mathcal{M} \to \mathcal{M}\ , \quad \text{similar to the uncoupled map}$$

$$\frac{\partial}{\partial x_j} F_i\ , \quad \text{small if } |i-j| \text{ large (more later)}$$

The main goal of this talk

- Describe a functional framework to study the properties of these maps
- Study hyperbolic sets:

  Their stability properties

  Invariant manifolds

Other problems can be studied with similar formalisms

- SRB measures and depends on parameters (M. Jiang, R.L.)
- KAM theory for whiskered tori (E. Fontich, Y. Sire, R.L.)
- “Bursting” (Blazevski, R.L.)
- Several other topics are in progress

Main references:

- E. Fontich, R. de la Llave, P. Martín

  http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=10-76

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1. Formalism

We would like to develop a formalism so slick that these systems become as similar as possible to finite dimensional systems.

This is impossible because these systems have uncountably many periodic orbits.

Hence, either compactness or differentiability have to be given up.

The arguments which do generalize to any setting have to be carefully chosen.

One possibility (natural from the point of view of Statistical Mechanics) is to consider finite systems and take the limit as the size grows.

We will perform analysis for the *whole infinite* system.

(Once we know how to make sense of the ∞ system, the convergence of finite approximations to the ∞ system is easier.)

We will give up compactness properties, but we will keep good differentiability properties.

Some carefully chosen proofs familiar from finite dimensional theory can be extended to the infinite dimensional setting.

Topology of $\mathcal{M}$

- We will model $\mathcal{M}$ on ℓ^∞

$$d(x,y) = \max d(x_i, y_i)$$

$\mathcal{M}$ is a Banach manifold modeled on

$$\ell^\infty(\mathbb{Z}^d, \mathbb{R}^n)$$

The reason to choose ℓ^∞ is that we do not want to restrict ourselves to properties that decrease at ∞.

Note

- We do have compactness under Tichonov's topology of coordinatewise convergence.
- Analysis in ℓ^∞ has peculiarities. There are functions which are not given by their matrix elements

  Ex: Banach limits in $\ell^\infty(\mathbb{Z}, \mathbb{R})$.

  Define $A(u) = \lim_{i\to\infty} u_i$ on a subspace.

  Use Hann-Banach theorem to extend $\partial_{u_i} A(u) = 0$.

  These "functionals at ∞" have the interpretation of "boundary conditions at ∞," closely related to phase transitions.

Our next goal is to define a norm on operators which are well behaved under compositions.

We say $\Gamma : \mathbb{Z}^d \to \mathbb{R}^+$ is a "decay" function

$$\sum_i \Gamma(i) \leq 1$$

$$\sum_j \Gamma(i-j)\Gamma(j-k) \leq \Gamma(i-k)$$

There is a nice physical interpretation of the condition on Γ.

If $\Gamma(i-k)$ is a bound on the direct interaction of particle i on particle j $\sum \Gamma(i-j)\Gamma(j-k)$ is a bound on the influence of particle i on particle j through indirect interactions on particles.

It is interesting to note

$\Gamma(i-j) = c\exp(-\theta|i-j|)$ is not a decay function

$$\Gamma(j-j) = c|i-j|^{-\alpha}\exp(-\theta|i-j|) \quad \alpha > d+1, \quad \theta \geq 0$$

is a decay function.

□ □ □

Define

$$|A_{ij}| = \sup_{\substack{\|u\|\leq 1\ ,\\ \pi_\ell u=0\ ,\\ \ell\neq j}} |\pi_i(Au)|$$

$$\gamma(A) = \sup_{ij} |A_{ij}|\Gamma^{-1}(i-j)$$

$$\|A\|_\Gamma = \max(\|A\|,\ \gamma(A))$$

Note that this definition allows boundary conditions at ∞, but the local interactions have to be "localized."

A crucial result is

$$\|AB\|_\Gamma \leq \|A\|_\Gamma \|B\|_\Gamma$$

So that the operators in this class are a Banach algebra.

□ □ □

One can also construct analogues of multilinear functions, etc.

One can use this to define C^k_Γ functions.

$$f : \mathcal{M} \to \mathcal{M} \ \text{ s.t. } \ \forall\, \ell \subset k$$

$$\sup_x \|D^k f(x)\|_\Gamma < \infty \, , \quad D^\ell f(x) \ \text{ continuous}$$

This is a Banach space.

We can also define spaces of C^α_Γ functions

$$|\pi_i[f(x) - f(y)]| \le \Gamma(i - j)|x - y|^\alpha$$

when x, y differ only on j coordinate.

These spaces satisfy many properties analogous to finite dimensional differentiable functions.

- Ascoli-Arzelá type theorems
- Hadamard-Kolmogorov interpolation
- Properties of composition

$$f, h \to f \circ h \text{ is differentiable}$$

$$C^3_\Gamma \times C^\alpha_\Gamma \to C^\alpha_\Gamma \qquad (D_h f \circ h)\eta = Df \circ h\eta$$

Note that, even in finite dimensions, to get that $D_h f \circ h$ maps C^α to itself we need that $f \in C^{1+\mathrm{Lip}}$. In finite dimensions, it suffices that $f \in C^{2+\varepsilon}$.

In the framework we have set up we have not included Hölder spaces with derivatives. It seems that introducing fractional derivatives, one can sharpen the results.

As a consequence of this framework

Theorem 1

Assume that $\Lambda \subset M$ *is a uniformly hyperbolic set for* $f : M \to M$, $f \in C^3$. *(Hence* $\Lambda^{\mathbb{Z}^d}$ *is hyperbolic for* $f^{\mathbb{Z}^d}$*.)*

$$g,\ C^3_\Gamma\text{-close to } f^{\mathbb{Z}^d}$$

Then, g has a hyperbolic set $\widetilde{\Lambda}$

$$g \circ h = h \circ f \mid h \in C^\alpha_\Gamma(\Lambda^{\mathbb{Z}^d})$$

h depends smoothly on parameters when g depends smoothly on parameters.

There is a hyperbolic splitting E^s_x, E^u_x, $x \in h(\Lambda)$, π^s_x, π^u_x *are* C^α_Γ.

Sketch

Inspired by Moser's proof of Anosov's structural stability, we consider

$$g \circ h - h \circ f = \mathcal{F}(g, h)$$

and apply Implicit Function Theorem (note that h^{-1}, considered by Moser originally, does not depend smoothly on parameters).

Showing h^{-1} exists, $h(\Lambda)$ is hyperbolic requires a bit more work.

Note that h^{-1} does not depend smoothly on parameters as examples show. This is what was used in Moser's original proof and work immediately after.

In practice, it is much best to use the "*shadowing argument*".

In the case that the maps are Anosov, [JiangLlave00] also showed that SRB measures for the ∞ system depend smoothly on parameters.

In this case, the definition of SRB measures that we use is the Gibbs equilibria for the unstable determinant (following the definition of [JiangPesin98]). Using h and the dependence of $E^u_{h(x)}$ one gets that the potential changes smoothly on parameters, but that the dynamics is not change. Note that other characterizations of SRB measures do not make sense (e.g. approximation by periodic orbits) or are problematic (iterations of measures equivalent to Lebesgue).

In [JiangL03] explicit formulas for the response (i.e. the derivative of the invariant measure.

Other work [BonettoKupiainenLebowitz] have studied the smooth dependence of the projections of SRB measures for special maps.

We can also study the stable and unstable foliations

$$W_x^s, W_x^u \text{ are } C_\Gamma^{k-2} \text{ manifolds}$$

$$x \to W_x^s, x \to W_x^u \text{ are } C^\alpha \text{ mappings}$$

Some of the usual properties of the characterization of hyperbolicity in terms of the spectrum of F_* also carry through.

Stability of spectrum, invariance under rotations of the spectrum.

One can give explicit values for the smallness constant in terms of the second derivatives of the perturbation and the norms of the inverse $F_* - \mathrm{Id}$. This norm can be expressed in terms of the hyperbolicity constants (angle between the stable and unstable splitings, exponential contraction rates, constants in expansion).

The parameterization method to study invariant manifolds

Given a map $F : X \to X$; $F(0) = 0$. We try to find $K : E \to X$, $P : E \to E$ polynomial

$$F \circ K = K \circ P$$

$K(0) = 0$ choice of the point

$$DF(0)DK(0) = DK(0)DP(0)$$

Choice of an invariant space for $DF(0)$.

One can match the equations for the expansion of K provided non-resonance conditions are met

$$F \circ (K^{\leq} + K^{>}) = K^{\leq} \circ P + K^{>} \circ P$$

One can use the fact that $K^{>}$ vanishes to high order to get a functional equation that can be solved by contraction.

Other results by similar methods

Quasi-periodic and almost periodic whiskered breathers in Hamiltonian coupled map lattices and flows (with E. Fontich, Y. Sire)

One can extend the formalism to study embeddings

$$K : \left(\prod^{d}\right)^{N} \to \mathcal{M} \quad \{c_j\}_{j=1}^{N} \subset \mathbb{Z}^d$$

$$|\partial_{\theta_j} K_i| \leq \max \Gamma(i - c_j)$$

The study of whiskered tori is based on the study of the equation

$$F \circ K(\theta) = F(\theta + \omega)$$

We assume $F \in C^{\omega}_{\Gamma}$ and preserve the symplectic form $\Omega^{\mathbb{Z}^d}$.

We do not assume F is close to integrable.

Assuming that ω Diophantine, K satisfies

- $\|F \circ K - K \circ T_\omega\|_{C^\omega_{\Gamma,c}} \leq \varepsilon$
- $DF \circ K$ admits a hyperbolic splitting

$$DF \circ K(\theta) E^{c,s,u}_\theta = E^{c,s,u}_{\theta+\omega}$$

$$\|DF \circ K|_{E^s}\| \leq \lambda < 1 \,, \quad \|DF^{-1} \circ K|_{E^u}\| \leq \lambda < 1$$

$$\|DF \circ K|_{E^c}\| \leq \eta \,, \quad \|DF^{-1} \circ K|_{E^c}\| \leq \eta \,, \quad \lambda\eta < 1$$

- $\dim E^c = 2Nd$
- The embedding satisfies some non-degeneracy condition
- ε sufficiently small

Then, $\exists\, \widetilde{K} \in C^{\omega,\delta'}_{\Gamma,c}$

$$\|K - K^*\|_{C^{\omega,\delta'}_{\Gamma,c}} \leq C\varepsilon$$

Two consequences

- The uncoupled solutions consisting of 1-KAM torus and all the other sites at the hyperbolic point persist.
- If the system is translation invariant we can build more complicated solutions out of simpler ones. By iterating the construction, one gets solutions with infinitely many frequencies.

Melnikov Theory

If the hyperbolic point of f_0 has homoclinic connections, one can construct homoclinic solutions for whiskered tori.

Normally hyperbolic invariant manifolds (work in progress)

One can develop a theory of normally hyperbolic invariant manifolds with decay.