Escape rates, entropy and Lyapunov exponents in dynamical systems with holes

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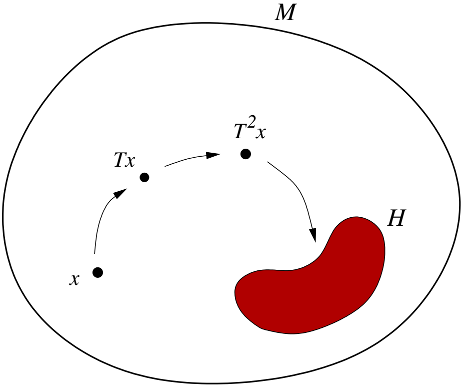
A dynamical system with a hole

$T : M \circlearrowleft$ a dynamical system
$H \subset M$ a "hole"
$\mathring{M} = M \backslash H$
$\mathring{T} = T|_{\mathring{M} \cap T^{-1}\mathring{M}}$ a *dynamical system with a hole*



Motivating questions

Q1 **Is the escape rate well-defined?** Starting with an initial distribution μ_0, does the limit

$$\rho(\mu_0) := \lim_{n\to\infty} \frac{1}{n} \log \mu_0\Big(\bigcap_{i=0}^{n} T^{-i}(M \backslash H) \Big) \quad \text{exist?}$$

We write $\overline{\rho}$ and $\underline{\rho}$ for the limsup and liminf.

Q2 Suppose the answer to Q1 is yes. **How does $\rho(\mu_0)$ vary with the size and position of the hole?**

Q3 The survivor set is the set of points in M that never enter H. **Does the escape rate correspond to a notion of pressure with respect to a class of invariant measures on the survivor set?**

Pressure and escape: Known results I

- Define the survivor set, $\Omega = \bigcap_{n=-\infty}^{\infty} T^{-n}(M \backslash H)$
- Let $\mathcal{E} = \mathcal{E}(\Omega) = \{$ergodic invariant measures supported on $\Omega\}$.
- Define $\mathcal{P}_{\mathcal{E}} = \sup_{\nu \in \mathcal{E}} P_\nu$ and $P_\nu = h_\nu(T) - \lambda_\nu^+$

  where $h_\nu(T) =$ metric entropy and λ_ν^+ is the sum of positive Lyapunov exponents at ν-a.e. point, counted with multiplicity.

Known results in this direction.

- (Bowen '75) If $T : M \circlearrowleft$ is a C^2 Axiom A diffeomorphism and $\Lambda \subset M$ is a *basic set*, then $\mathcal{P}_{\mathcal{E}(\Lambda)} \le 0$ and $\mathcal{P}_{\mathcal{E}(\Lambda)} = 0$ iff Λ is an attractor.
- (Young '90) $T : M \circlearrowleft$, C^2 diffeomorphism of a compact Riemannian manifold; m Lebesgue measure. Let $H \subset M$ be open and assume
  - (i) Ω is compact with $d(\Omega, \partial H) > 0$ and
  - (ii) $T|_\Omega$ uniformly hyperbolic.

  The $\rho(m)$ is well defined and equals $\mathcal{P}_{\mathcal{E}}(\Omega)$.

Pressure and escape: Known results II

For **small** holes in more general situations, an escape rate formula has been proved:

$$\exists\, \nu \in \mathcal{E} \text{ such that } \rho(m) = h_\nu(T) - \lambda_\nu^+ .$$

- Expanding maps in $\mathbb{R}^n$ with a finite Markov partition after the introduction of a hole [Collet, Martínez, Schmitt '94]
- C^2 Anosov diffeomorphisms [Chernov, Markarian, Troubetzkoy '97-'00]
- Collet-Eckmann and piecewise expanding maps of the interval [Bruin, Melbourne, D '10]

All these results view the map with a small hole as a perturbation of the map without a hole, prove the persistence of a spectral gap of the transfer operator and use it to construct such a measure ν.

Variational inequality for smooth systems with holes

$T : M \circlearrowleft C^2$ diffeomorphism of compact, Riemannian manifold
$H \subset M$ open set. $m =$ Lebesgue measure.
For $\varphi \in L^1(m)$, let $m_\varphi = \varphi m$. Define

- $\mathcal{G}_H = \{\nu \in \mathcal{E} :$ there exist $C, \alpha > 0$ such that $\nu(N_\varepsilon(\partial H)) \le C\varepsilon^\alpha$ for all $\varepsilon > 0\}$,
- $\mathcal{G}_\varphi = \{\nu \in \mathcal{E} : \exists \delta > 0$ and an open set V such that $\nu(V) > 0$ and $\varphi|_V > 0\}$.

Theorem (Variational inequality for absolutely cont. measures)

$$\underline{\rho}(m_\varphi) \ge \mathcal{P}_{\mathcal{G}_H \cap \mathcal{G}_\varphi} = \sup_{\nu \in \mathcal{G}_H \cap \mathcal{G}_\varphi} \{h_\nu(T) - \lambda_\nu^+\}$$

Note: If $\nu \notin \mathcal{G}_\varphi$, then ν cannot "see" escape with respect to m_φ. Also, if $\varphi \ge c > 0$, then $\mathcal{G}_\varphi = \mathcal{E}$.

Variational inequality for systems with singularities

Singularity Set $\mathcal{S}$. T is C^2 on $M \backslash \mathcal{S}$.
Assume there exist constants $C, a > 0$ such that

$$\|DT_x\| \leq Cd(x, \mathcal{S})^{-a} \qquad \text{and} \qquad \|DT_x^{-1}\| \leq Cd(x, T\mathcal{S})^{-a}$$

Similar condition on $D^2 T_x$. Define

- $\mathcal{G}_\mathcal{S} = \{\nu \in \mathcal{E} :$ there exist $C, \alpha > 0$ such that $\nu(N_\varepsilon(\mathcal{S})) \leq C\varepsilon^\alpha$ for all $\varepsilon > 0\}$.

Theorem (Variational inequality)

Let $m_\varphi = \varphi m$ be as before, then

$$\underline{\rho}(m_\varphi) \geq \mathcal{P}_{\mathcal{G}_H \cap \mathcal{G}_\varphi \cap \mathcal{G}_\mathcal{S}} = \sup_{\nu \in \mathcal{G}_H \cap \mathcal{G}_\varphi \cap \mathcal{G}_\mathcal{S}} \left\{ h_\nu(T) - \lambda_\nu^+ \right\}$$

SRB measures as initial distributions

Assume $T : M \circlearrowleft$ has no zero Lyapunov exponents and $\mu_{\mathsf{SRB}} =$ unique SRB measure for T
(abs. cont. conditional measures on unstable manifolds)

There exists a sequence of closed sets $\Sigma_1 \subset \Sigma_2 \subset \ldots$ with $\mu_{\mathsf{SRB}}(\cup_\ell \Sigma_\ell) = 1$ and $T|\Sigma_\ell$ has uniform hyperbolic properties [Pesin '77, Katok Strelcyn '86].

- $\mathcal{G}_{\mathsf{SRB}} = \{\nu \in \mathcal{E} : \nu(\Sigma_\ell) > 0 \text{ for some } \ell > 0\}$

Theorem (Variational inequality with respect to SRB)

$$\underline{\rho}(\mu_{\mathsf{SRB}}) \geq \mathcal{P}_{\mathcal{G}_H \cap \mathcal{G}_{\mathcal{S}} \cap \mathcal{G}_{\mathsf{SRB}}} = \sup_{\nu \in \mathcal{G}_H \cap \mathcal{G}_{\mathcal{S}} \cap \mathcal{G}_{\mathsf{SRB}}} \{h_\nu(T) - \lambda_\nu^+\}$$

Idea of proof for variational inequality

Use volume estimates on dynamically defined balls in M.

$$g_\varepsilon(x) = \frac{1}{3}\min\{\varepsilon, d(x, \partial H \cup \mathcal{S})\}$$

$$B(x, n, g_\varepsilon) = \{y \in M : d(T^i x, T^i y) < g_\varepsilon(T^i x), 0 \le i \le n\}$$

Write $M^n := \cap_{i=0}^n T^{-i}(M \backslash H)$ as the union of balls $B(x, n, g_\varepsilon)$.

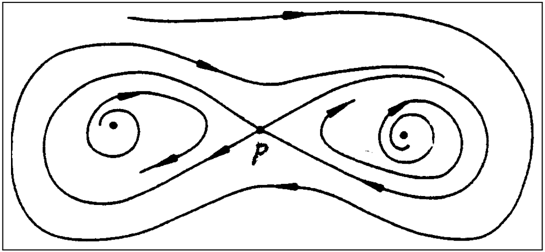
- $\nu(B(x, n, g_\varepsilon)) \approx e^{-nh_\nu(T)}$ ([Mañé '81], [Brin, Katok '81]), so there are $\approx e^{nh_\nu(T)}$ such balls which cover M^n.
- Each ball has volume $m(B(x, n, g_\varepsilon)) \gtrsim e^{-n\lambda_\nu^+}$. This volume estimate uses the Lyapunov charts associated with T.

Key fact: The ball centered at almost every point in M is never cut by $\partial H \cup \mathcal{S}$.

Full variational principle

Q: Under what conditions can we obtain the reverse inequality, i.e., a full variational principle?

Example: Figure 8 attractor.



Let $M \backslash H$ contain a neighborhood of the homoclinic orbit of p, a hyperbolic fixed point. Then δ_p is the only invariant measure in $\mathcal{E}$, so $\mathcal{P}_{\mathcal{E}} < 0$, but $\rho(m) = 0$.

Example I: Periodic Lorentz gas with holes I

- Billiard table created by removing finitely many scatterers from $\mathbb{T}^2$
- Billiard flow is given by a point particle moving at unit speed with elastic collisions at the boundary

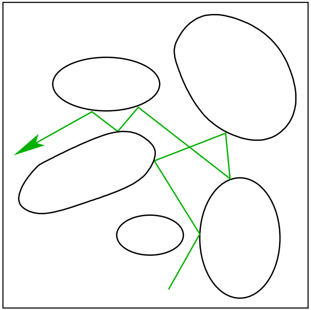


Figure: A billiard table

$M = \left(\cup_i \, \partial \Gamma_i \right) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow.

$T : M \circlearrowleft$ is the first return map, i.e. the *billiard map*.

Example I: Periodic Lorentz gas with holes II

Some Classical Facts about the Periodic Lorentz gas

Assume the scatterers have strictly positive curvature and satisfy the finite horizon condition.

- Tangential collisions with scatterers create singularity curves for T where DT blows up
- T is uniformly hyperbolic away from singularity curves
- T is ergodic and mixing, with exponential decay of correlations
- Liouville measure for the flow projects to a natural T-invariant, smooth SRB measure on M.

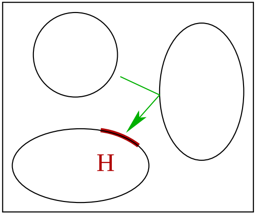
Results due to [Sinai '70], [Young '98].

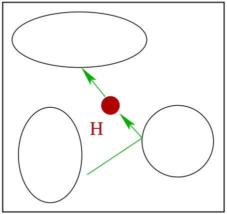
Example I: Periodic Lorentz gas with holes III

Introduction of holes

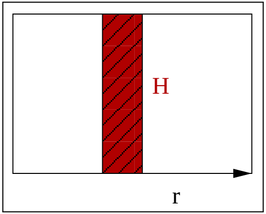
Type I Hole | Type II Hole

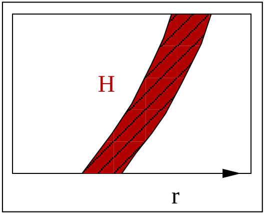
Billiard table





Phase Space





A hole of Type II for the flow and for the map

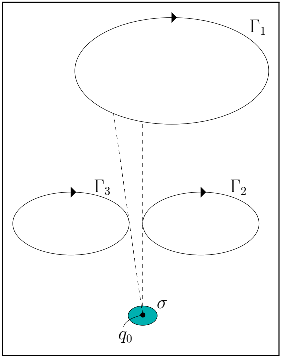


Figure: Type II hole

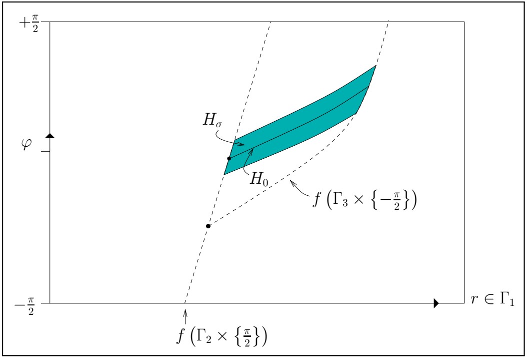


Figure: The induced hole for T

Key Fact: ∂H is always transverse to the unstable cone for T.

Results for the Lorentz Gas with holes I

Theorem

Suppose H is a small hole of Type I or II. Then $\rho(\mu_{\text{SRB}})$ is well-defined and

(a) $\mathring{T}_*^n \mu_{\text{SRB}} / |\mathring{T}_*^n \mu_{\text{SRB}}|$ converges to a conditionally invariant distribution μ_∞, $\mathring{T}_* \mu_\infty = e^\rho \, \mu_\infty$.

(b) μ_∞ is singular with strictly positive absolutely continuous conditional densities on unstable leaves in $M \setminus \cup_{i=0}^\infty T^i H$.

(c) Define

$$\nu(f) = \lim_{n \to \infty} e^{-n\rho} \int_{M^n} f \, d\mu_\infty \qquad \text{for continuous } f.$$

Then ν is ergodic with exponential decay of correlations, and

$$\rho(\mu_{\text{SRB}}) = P_\nu = \mathcal{P}_{\mathcal{G}_H \cap \mathcal{G}_{\mathcal{S}} \cap \mathcal{G}_{\text{SRB}}}.$$

Generalization: Systems admitting towers

Two key facts about the Lorentz gas which we can use to generalize this example:
(1) T is piecewise hyperbolic
(2) T admits a Markov extension $F : \Delta \circlearrowleft$ in the form of a Young tower which respects the hole.

- $F : \Delta \circlearrowleft$ has a countable Markov partition
- $\pi^{-1}H$ is a union of Markov partition elements
- Schematically, Δ is a tower: One base state to which all others return
- The base of the tower is constructed on a stack of unstable leaves with uniformly hyperbolic properties

In this setting, when the tail of the tower decays exponentially fast, we can formulate a theorem similar to the one stated for the Lorentz gas with small holes.

Example II: Anosov diffeomorphisms

Recall

- $\mathcal{E} = \mathcal{E}(H) = \{$ergodic invariant measures supported on $\Omega\}$.
- $\mathcal{G}_H$ are measures in $\mathcal{E}$ whose support is not concentrated near ∂H.
- $\overline{\rho}(m) = \limsup_{n\to\infty} \frac{1}{n} \log m(M_0^n)$, $\quad \underline{\rho}(m) = \liminf_{n\to\infty} \frac{1}{n} \log m(M_0^n)$.

Theorem (Variational inequalities for Anosov diffeomorphisms)

Let $H \subset M$ be open with finitely many simply connected components and ∂H compact.

1. $\mathcal{P}_{\mathcal{G}_H} \le \underline{\rho}(m) \le \overline{\rho}(m) \le \mathcal{P}_{\mathcal{E}}$.
2. If $d(\Omega, \partial H) > 0$, then $\mathcal{E} = \mathcal{G}_H$ so that the escape rate is well-defined and $\mathcal{P}_{\mathcal{G}_H} = \rho(m) = \mathcal{P}_{\mathcal{E}}$.
3. Counterexamples exist for which the inequalities are strict

The escape rate as a function of the hole

Suppose $\{H_\gamma\}_{\gamma \geq 0}$ continuous nested family of holes, $H_0 =$ point.

Small holes: $\gamma \mapsto \rho_\gamma$ is continuous.
If $d(H_\gamma, H_{\gamma'}) \leq C|\gamma - \gamma'|$, then
$|\rho_\gamma(m) - \rho_{\gamma'}(m)| \leq C'|\gamma - \gamma'| \log|\gamma - \gamma'|$. [D, Liverani '08]

Large or medium-sized holes: $\gamma \mapsto \rho_\gamma$ can have jumps and in general is neither upper nor lower semi-continuous.

Along typical sequences, the graph of $\gamma \mapsto \rho_\gamma$ forms a **devil's staircase** with jumps

- If $d(\Omega(H), \partial H) > 0$, then $\Omega(H) = \Omega(H')$ for H' sufficiently close to H. So $\mathcal{P}_{\mathcal{E}}(H) = \mathcal{P}_{\mathcal{E}}(H')$ and the escape rate is locally constant.
- When $\Omega \cap \partial H \neq \emptyset$, there can exist periodic points in Ω which cause the variational inequalities to be strict. Moving the boundary of the hole across one of these periodic points can cause the escape rate to jump.