Subdiffusive behaviour generated by some non-hyperbolic (but ergodic) systems

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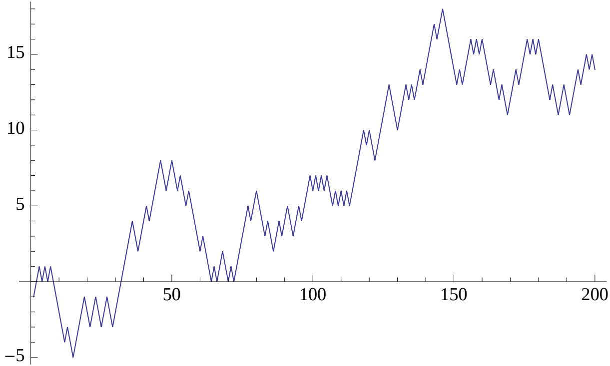

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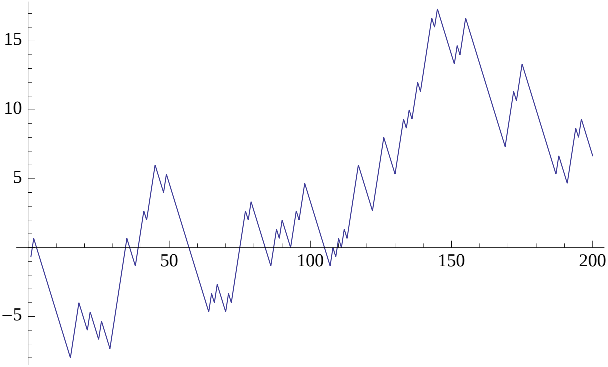

As before with $a = 2/3$ and $b = 4/3$

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More generally, slightly different behaviour for $f(x) = 2(\chi_{[0,\beta)}(x) - \beta)$ depending whether β is a q-adic rational or not.

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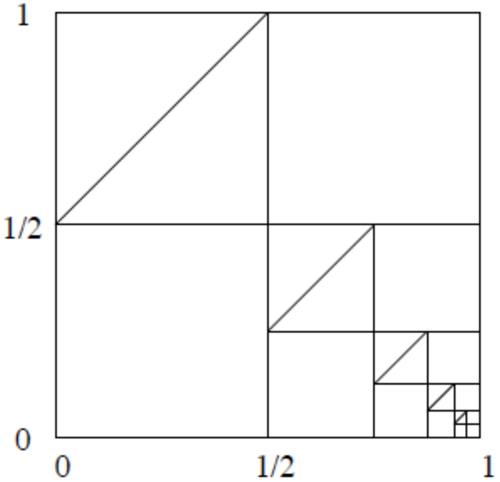

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$$T \circ \Psi(z) = \Psi \circ U(z) \quad , \quad \forall z \in \mathbb{Z}_q$$

Using this fact, one shows that T acts as a one cycle permutation on the set $\left\{ \left[\ell/q^k, (\ell+1)/q^k\right), 0 \le \ell < q^k \right\}$ of q-adic intervals of length q^{-k}, Using this fact, one shows that T acts as a one cycle permutation on the set $\left\{ \left[\ell/q^k, (\ell+1)/q^k\right), 0 \leq \ell < q^k \right\}$ of q-adic intervals of length q^{-k}, and $d(x, T^{q^k}(x)) < q^{-k}$ for all $x \in [0, 1)$ (all points are positively recurrent).

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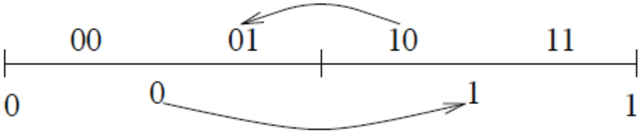

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upper bound:

$$\|S_n(f)\|_\infty \leq (q-1)V(f)(1 + \log_q n)$$

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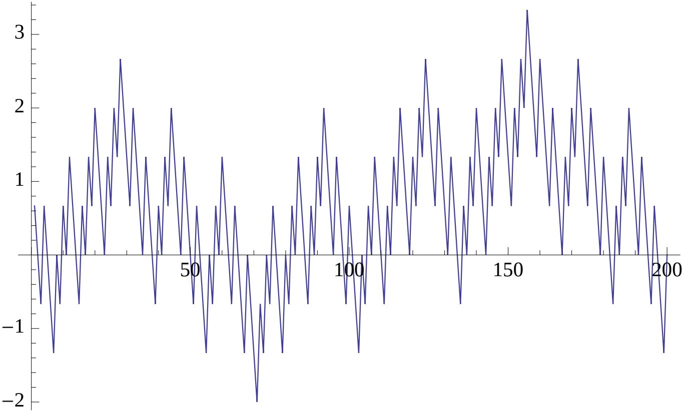

$q = 2$ and $\beta = 1/3$

$$\|S_{n_k}(f)\|_\infty \asymp \log_q n_k$$

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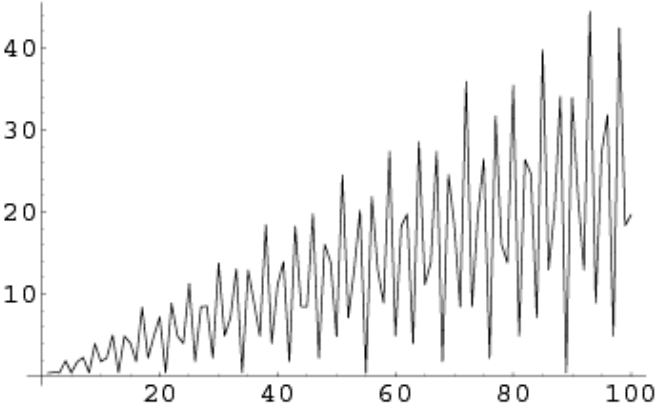
Example: $a_i = 2^{2^i} \Rightarrow \gamma = 2$.



$r \cdot \|r\alpha\|$ vs r for $\alpha = (\sqrt{5}-1)/2$

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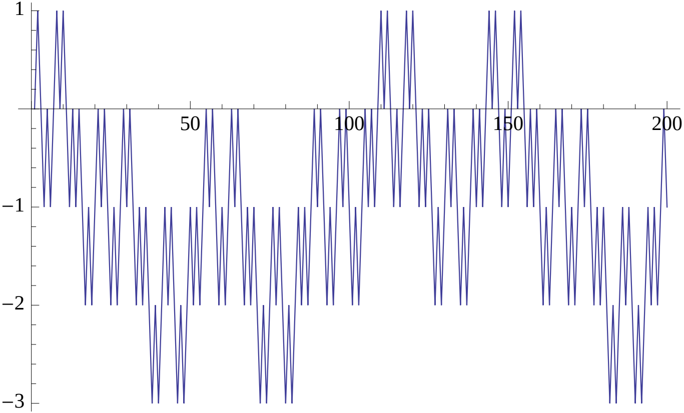

S_n vs n for $\alpha = (\sqrt{5}-1)/2$, with $a_i = 1$, $\forall i \geq 1$, and $f(x) = 2\chi_{[0,1/2)}(x) - 1$

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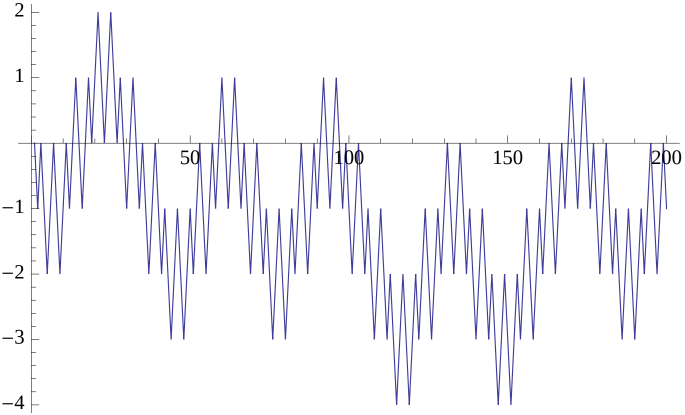

S_n vs n for $\alpha = e - 2$, with $a_i = 2l$ for $i = 3l - 1$, $l \geq 1$, and $a_i = 1$ otherwise (f as before).

$$\|S_n(f, \alpha)\|_\infty = O(\log^2 n / \log^2 \log n)$$

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Note: this cannot be applied if α is of type 1 and the Fourier coefficients f_r decay as (or faster than) $1/r$.

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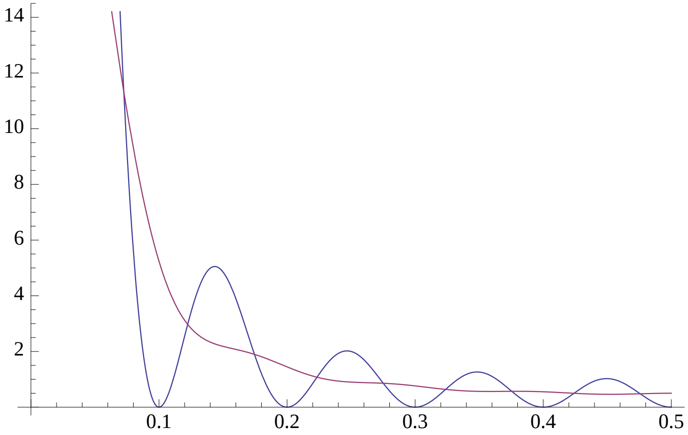

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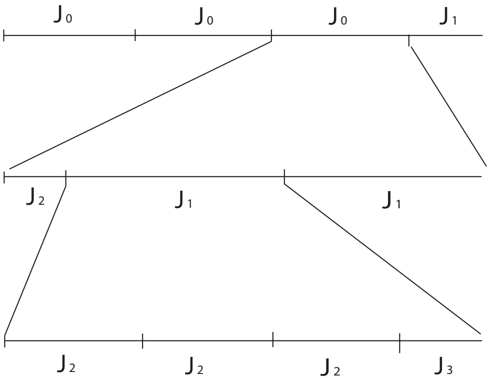
Diffusion and discrepancy via renormalization

Consider again the sequence of successive closest distances to the initial point $d_k := \|q_k\alpha\| = (-1)^k(q_k\alpha - p_k)$. We have

$$d_0 = \alpha, \quad d_1 = 1 - a_1\alpha, \quad d_2 = \alpha - a_2(1 - a_1\alpha), \ \ldots$$

which can be associated to a family of nested arcs J_k:



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- Three distance theorem: the sequence $\{r\alpha\}$ with $0 \leq r < n$ partitions the circle into n intervals whose lengths are $\ell_1 = d_k$, $\ell_2 = d_{k-1} - j\, d_k$ for some k and $1 \leq j \leq a_{k+1}$, and $\ell_3 = \ell_1 + \ell_2$ (which may disappear).

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Lemma. Setting $r_m := \min\{r \geq 0 \,:\, [r\alpha] = m\}$ we have

$$t_m := \#\{r \geq 0 \,:\, [r\alpha] = m\} = \begin{cases} a_1 + 1 & \text{if } \{r_m\alpha\} < d_1 \\ a_1 & \text{otherwise} \end{cases}$$

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We can then restrict to study what happens for $r_{m_j} \leq r \leq r_{m_j} + a_1$ with $t_{m_j} = a_1 + 1$, and this can be done by looking at the first return map on the interval $J_1 = [0, d_1)$, which is isomorphic to the rotation $\tilde{T}$ on X through the angle $\tilde{\alpha} = d_2/d_1 = G^2(\alpha)$.

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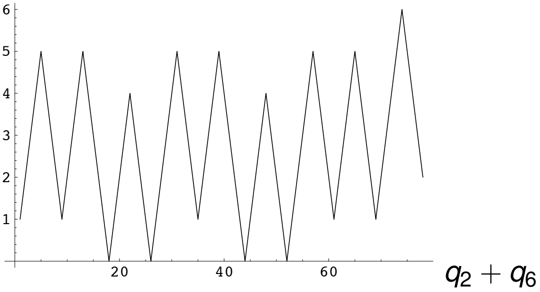
In a similar (but somewhat more involved) way one constructs $\mathcal{R}_{\mathrm{odd}}$ corresponding to a_1 odd.

$$S_n(f,\alpha), \quad \alpha = (7 + \frac{2}{\sqrt{5}-1})^{-1} = [8,1,1,1,\dots]$$



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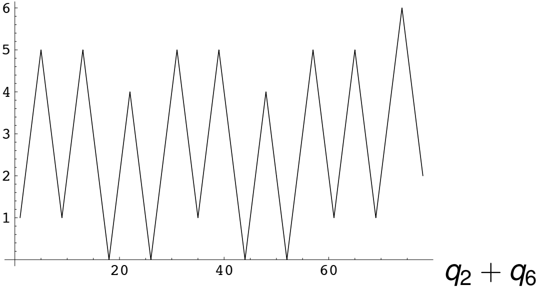

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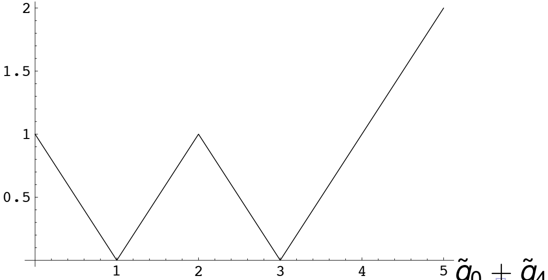

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Theorem. Let a_{2i+1} be even $\forall i \geq 0$ and $r = \sum_{i=2}^{N} c_i q_i$ then

$$\frac{1}{2}\sum_{i=0}^{\frac{N}{2}-1} a_{2i+1} \leq \max_{0\leq n\leq r} S_n(f,\alpha) \leq \frac{N}{2} + \frac{1}{2}\sum_{i=0}^{\frac{N}{2}-1} a_{2i+1}$$

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$$\limsup_{r\to\infty} \frac{\max_{0\leq n\leq r} S_n(f,\frac{\sqrt{5}-1}{2})}{\log r} \leq \frac{1}{6\log\left(\frac{\sqrt{5}+1}{2}\right)}$$

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Theorem. Let α have unbounded partial quotients and denote ν_{even} and ν_{odd} the limits

$$\nu_* := \liminf_{k\to\infty} \frac{\sum_{i=1}^{k}{}^{*} a_i}{\sum_{i=1}^{k} a_i}$$

then

$$\frac{1}{4}\max\{\nu_{\mathrm{even}}, \nu_{\mathrm{odd}}\} \le \limsup_{n\to\infty} \frac{n\, D_n^*(\alpha)}{\sum_{i=1}^{N} a_i} \le \frac{1}{4}$$

where $n = \sum_{i=0}^{N} c_i q_i$.