Some Examples of Hyperbolic Dynamics

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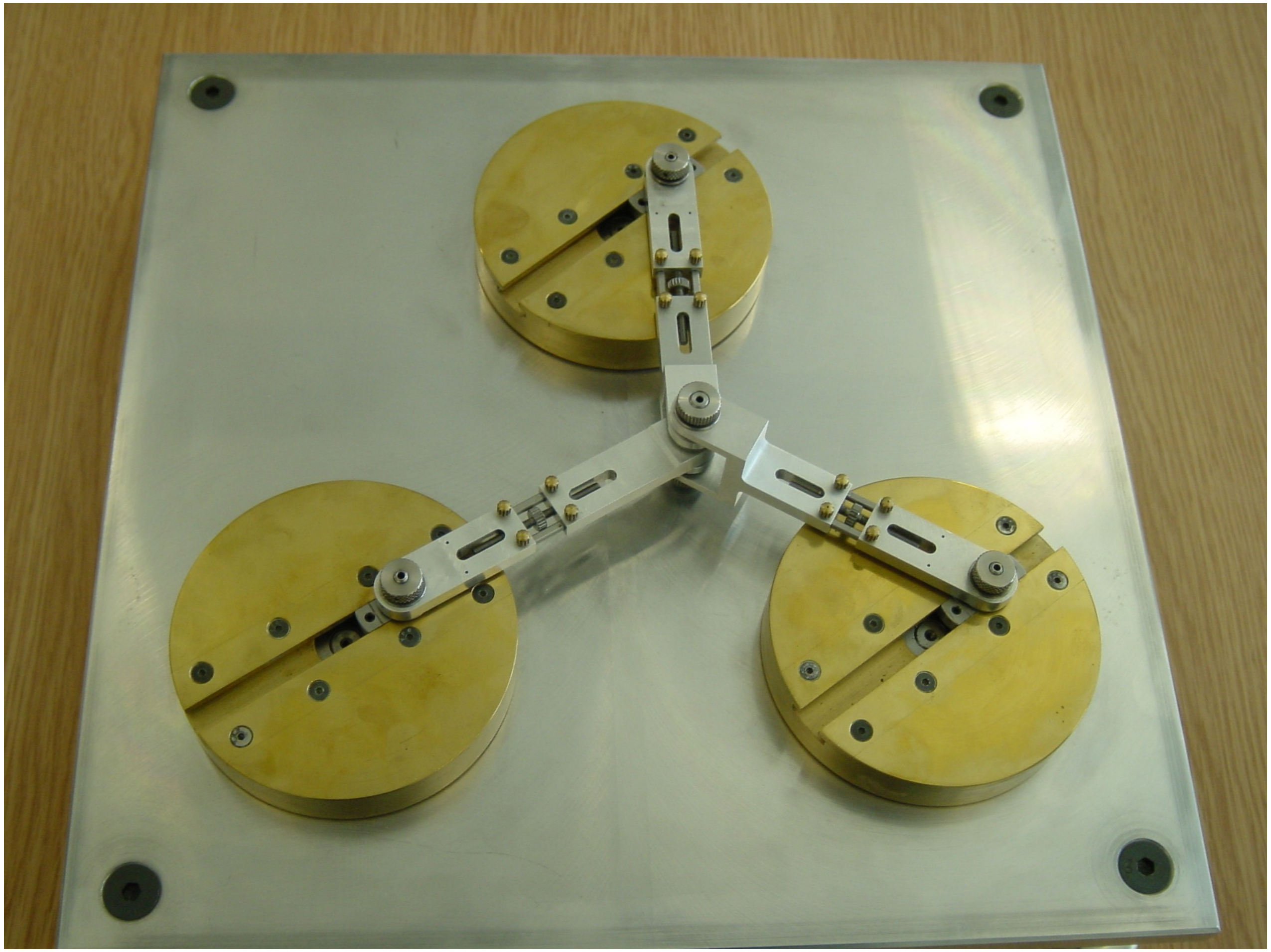
Outline

1. Triple Linkage
2. Steady mixing volume-preserving flows
3. Cerbelli-Giona map
4. Geodesics in de Sitter space

1. Triple linkage

- Anosov systems = the whole state space is uniformly hyperbolic (splitting of tangent bundle into forwards and backwards contracting subbundles with uniform exponential bounds; equivalently, uniformly bounded Green function)
- The nicest form of chaos (Markov partition, Gibbsian dynamics)
- Mathematicians' favourite example: geodesic flow on a surface of negative curvature
- Physicists say they never see them.
- They didn't look in the right places.

The Triple Linkage



Show video

Its frictionless dynamics

- Is Anosov on every positive energy level, for suitable range of parameters
- Because (up to rescaling time) it is a small pertur-bation of geodesic flow on Schwartz P-surface $\Sigma_i \cos(\theta_i) = 0$ with $ds^2 = \Sigma_i d\theta_i^2$, for which $\kappa = -\Sigma_i \cos^2(\theta_i)/\Sigma_i \sin^2(\theta_i) \leq 0$ with equality only if each $\theta_i = \pm\pi/2$ [cf. Knauf]
- Also mixing, because T_1X_3 does not fibre over S^1 [apply Anosov]
- Holder functions mix exponentially [apply Liverani, Dolgopyat]
- Brownian limit law in any Abelian cover [apply Melbourne&Nicol]
- Same if add a smooth potential, for all sufficiently high energies

Effects of friction and driving

- Every trajectory goes to a set of equilibria
- For friction directly opposing momentum (and no potential), solutions are pieces of geodesic in rescaled time, e.g. constant coefficient linear case: $\theta(C(1-e^{-\alpha t})/\alpha)$, $C = \sqrt{2E_0/I}$.
- Can add feedback control, e.g. torques $\Gamma_i = -\gamma(K-E)d\theta_i/dt$ with $\gamma>\sqrt{I/2E}$, and permit weak friction, to make an attracting invariant manifold near to unperturbed energy level K=E and the dynamics on it is Anosov.
- Any weak driving force which keeps system in $K \geq E$ and partially hyper-bolic there produces an attracting set with Lyapunov exponent $\geq k \sqrt{E}$ for every orbit.
- Add a constant torque (or put in a plane containing the vertical, wrap a string round one axle and hang a weight on it): unfortunately, looks like goes to an attracting periodic orbit.
- Ref: Hunt & MacKay, Nonlinearity (2003)

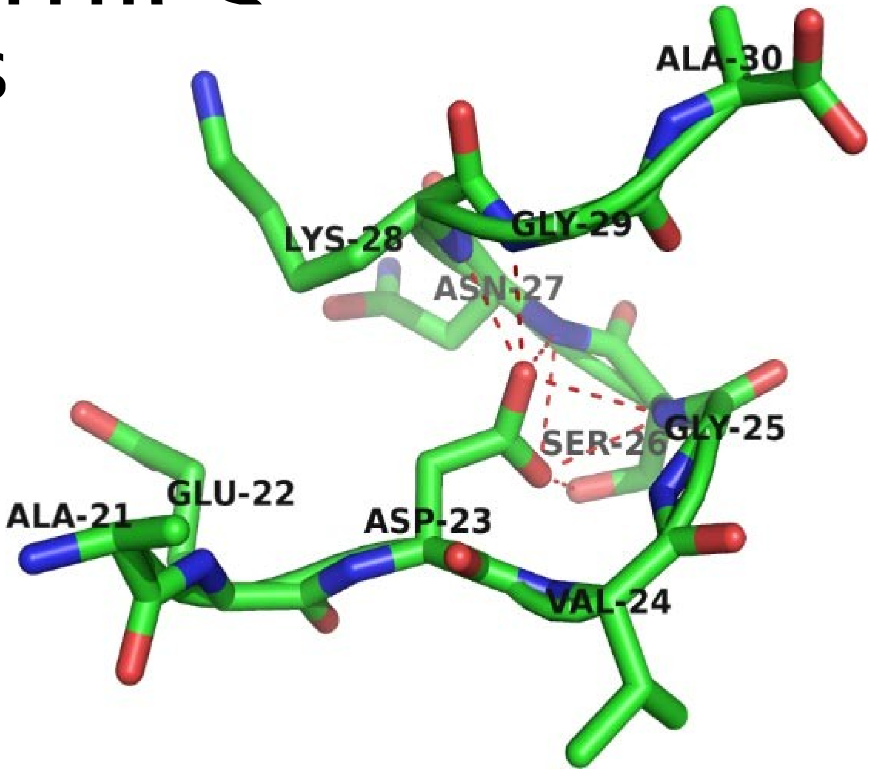
Adding degrees of freedom

- Could attach a slightly off-centre rotor to any moving part; resulting system is partially hyperbolic except where nearly all the energy is in the rotor; according to Pugh&Shub, expect frictionless dynamics to be ergodic on each energy level; it would be nice to compute a sufficient condition by first order perturbation theory
- Could couple two triple linkages by a weak spring: partially hyperbolic except where one has nearly zero energy; might hope to derive a stochastic equation for the relative energy and time-shift
- Higher dimensional Anosov systems? e.g. zigzag chain of disks connected by linkages

Real-world Linkage dynamics

ck to edit Master text s
Second level
Third level
· Fourth level
– Fifth level

Metris coordinate measuring robot



Amyloid-beta loop implicated in Alzheimer’s disease (D Teplow)

Chaos

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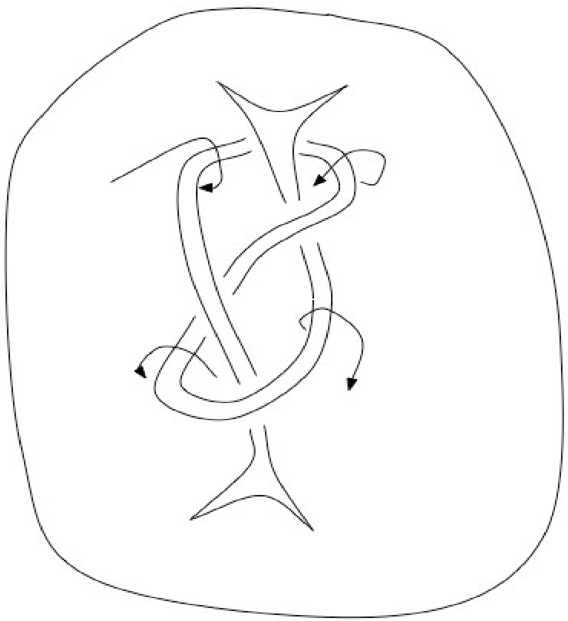
2. 3D Mixing flows

- Fluid mechanics began to realise that particle motion in steady 3D flow could be chaotic.
- But they never made a (perfectly) mixing flow
- Arnol’d dynamo flow (suspension of cat map) is mixing if roof function non-trivial, but not realisable in a container in R3.
- But could puncture along the “period 1” orbit…

Figure-eight knot flow

- Blowup along the "period 1" orbit gives an almost Anosov flow in a figure-eight knot complement
- Scale to make velocity go to 0 at boundary (no-slip)
- Given choice of figure-eight knot complement in R3 and velocity scaling, can choose roof function to preserve Euclidean volume
- In process of proving flow is mixing and structurally stable in C3 volume-preserving no-slip flows (with Ru)
- Tried to get Simone Cenedesi (Murano) to make me an example out of glass
- How to drive such a flow?



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Pigtail braid flow

- Alternatively, can blowup along period 1 and period 3 orbits, and quotient by reflection through the period 1 orbit, obtaining a mixing flow in a solid torus minus the closure of a pigtail braid.
- Could take a Z-cover; might Stokes' flow realise it (minimum dissipation rate)?
- Problem of how to hold the pigtail
- Analysis slightly different because 1-prong orbits instead of saddles
- Nontrivial limit law for dispersal along a mixing flow in a no-slip pipe: Gaussian but with width like sqrt{t log t} (with Kolokoltsov)

Baker's flow

- If willing to leave the almost Anosov world and make do with singular-hyperbolic systems then can make other mixing examples.

Also, periodic pipe versions

Ref: MacKay, in Chaos, Complexity and Transport, eds Chandre et al (2008)

3. Cerbelli-Giona map

- H: x’ = x + f(y), y’ = y+x’ on T2.
- Proposed as a toy model in which to study effects of mixing on chemical reactions
- They found invariant line fields with uniform contraction in +/- time, and by constructing the corresponding foliations proved mixing.

It is Pseudo-Anosov

- 4 singularities
- Markov partition
- Perron-Frobenius eigenvalue

  $\lambda = \frac{1}{2}(1+\sqrt{3}+\sqrt{(2\sqrt{3})})$
- Eigenvectors for transition matrix and its transpose generate the transverse measures $\mu\pm$

with
Gin

Some consequences

- Explicit calculation of multifractal characteristics, e.g. dimension of the set for which μ+ has pointwise dimension α.
- H2R, where R(x,y)=(x-1,y) on RxT, is an exact area-preserving tilt map, ergodic, positive metric entropy [log(2+√3)], and has Brownian limit [with diffusion 1/4√3]; conjecture it can be smoothed to a codimension-3 submanifold of C2 a-p maps with the same properties.
- Ref: MacKay, J Nonlin Sci (2006)

4. De Sitter space

- -x02 + Σi xi2 = a2 with ds2 = -dx02 + Σi dxi2, i=1..4
- Solution of Einstein vacuum equations with cosmological constant 3/a2.
- Its geodesics are the (components of) intersection with 2-planes through 0
- Timelike geodesic flow (on tangent vectors with ds2 = -1) is Anosov: unstable manifold of a unit tangent is given by the unit normals to its “expanding flat slice”, e.g. x0+x1 = a for unit tangent (1,0,**0**) at (0,a,**0**)

Moschella

Some consequences

- Same for all small perturbations, e.g. Schwarzschild-de Sitter more than (Ma2)1/3 away from black hole
- Hubble's law does not imply big bang: backwards converging trajectories of an Anosov system typically diverge further back
- Ref: MacKay and Rourke, in preparation

Conclusion

- There are many opportunities to apply hyperbolic dynamical systems theory
- Some others:

- Second species chaos in celestial mechanics (Bolotin & MacKay, Cel Mech 2000, 2006) and for charges in magnetic fields (Pinheiro & MacKay, 2006, 2008)
- Synchronisation of non-autonomous oscillators is existence of a normally hyperbolic cylinder in extended state space (with Gin)
- Transition states are normally hyperbolic submanifolds spanned by surfaces of locally minimal flux: study their bifurcations (with Strub)