Attractors in billiards with dominated splitting

We prove that trajectories in a huge class of "billiards´´ with angle of reflection different than angle of incidence have

dominated splitting: tangent bundle splits into two invariant directions, the contractive behavior on one of them dominates the other one by a uniform factor.

The three types of attractors predicted in the paper by Pujals and Sambarino (Annals of Math., 2009) appear in the dynamics of these billiards

A. Arroyo, R. Markarian, D. Sanders: *Bifurcations of periodic and chaotic attractors in pinball billiards with focusing boundaries* (Nonlinearity), UNAMexico

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Billiards: math. models for physical phenomena where hard balls move in a container with elastic collisions on its walls and/or with each other.

A point particle moves on Riem. manifold with boundaries. They determine dynamical props.

May vary from completely regular (integrable) to fully chaotic. Examples: dispersing billiard tables due to Ya. Sinai (model of hard balls studied by L. Boltzmann and the Lorentz gas).

In contrast, billiards in polygonal tables are not hyperbolic, but generically ergodic.

The dynamics of **classical** billiards are prototypes of conservative dynamics: the Liouville measure is preserved: they are not useful to model rich phenomena that could hold in regimes far from the equilibrium.

Non-elastic billiards:

The particle moves along straight lines inside the billiard table; it hits one of the walls with angle η with respect to the normal, it is reflected with angle ϕ.

If $\phi = \lambda\eta$ (with $\lambda \leq 1$): the ball is "kicked" by the wall giving a new impulse in the direction of the normal and thereby increasing its kinetic energy (*pinball billiards*)

Consider the diffeomorphism $f : M \to M' \subset M$, where M is a riemannian manifold. An f-invariant set Λ is said to have *dominated splitting* if we can decompose its tangent bundle in two invariant continuous subbundles $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{|E(x)}\| \, \|Df^{-n}_{|F(f^n(x))}\| \leq Ca^n, \text{ for all } x \in \Lambda, n \geq 0.$$

with $C > 0$ and $0 < a < 1$; *a is called a constant of domination*. It is assumed that neither of the subbundles is trivial (otherwise, the other one has a uniform hyperbolic behavior).

Any hyperbolic splitting is a dominated one.

Meaning of the above definition: it says that, for n large, the "greatest expansion" of Df^n on E is less than the "greatest contraction" of Df^n on F, and by a factor that becomes exponentially small with n.

In other words, every direction not belonging to E must converge exponentially fast under iteration of Df to the direction F.

Limit set: $L(f) = \overline{\bigcup_{x \in M} (\omega(x) \cup \alpha(x))}$ $x \in M$ is *nonwandering* with respect to f if for any open set containing x there is a $N > 0$ such that $f^N(U) \cap U \neq \emptyset$. Set of all nonwandering points of f is denoted by $\Omega(f)$. $B \subset M$ is called *transitive* if there exists a point $x \in B$ such that its orbit $\{f^n x\}_{n \in \mathbb{Z}}$ is dense in B

Compact invariant submanifold V is *normally hyperbolic* if the tangent space to the ambient space can decompose in three invariant continuous subbundles $T_V M = E^s \oplus TV \oplus E^u$, such that:

$$\inf_{x \in V} m(D_x f_{|E^u(x)}) > \sup_{x \in V} \|D_x f_{|TV(x)}\|,$$

$$\sup_{x \in V} \|D_x f_{|E^s(x)}\| < \inf_{x \in V} m(D_x f_{|TV(x)})$$

where the minimum norm $m(A)$ of a linear transformation A is defined by $m(A) = \inf\{\|Au\| : ||u|| = 1\}$.

Consequences of dominated splitting

One of the main goals in dynamics is to understand how the dynamics of the tangent map Df controls or determines the underlying dynamics of f.

Smale: if limit set $L(f)$ splits into invariant subbundles, $T_{L(f)}M = E^s \oplus E^u$ and vectors in E^s are contracted by positive iteration by Df (E^u, by negative iteration) $L(f)$ *can be decomposed into disjoint union of finitely compact maximal invariant and transitive sets; periodic points are dense in* $L(f)$*; asymptotic behavior of any trajectory is represented by an orbit in* $L(f)$*.*

A natural question arises: is it possible to describe the dynamics of a system having dominated splitting?

Moreover, since in dimension larger than two examples of open sets of non-hyperbolic diffeomorphisms that have a dominated splitting exist, it is natural to ask: under the assumption of dominated splitting, is it possible to conclude hyperbolicity in dimension two?

In fact, a similar spectral decomposition theorem as the one stated for hyperbolic dynamics holds for smooth surface diffeomorphisms exhibiting a dominated splitting.

Theorem (PS09) *Let* $f \in \mathrm{Diff}^2(M^2)$ *and assume that* $\mathrm{L}(f)$ *has a dominated splitting. Then* $\mathrm{L}(f)$ *can be decomposed into* $\mathrm{L}(f) = \mathcal{I} \cup \tilde{\mathcal{L}}(f) \cup \mathcal{R}$ *such that*

1. $\mathcal{I}$*, set of periodic points with bounded periods contained in a disjoint union of finitely many normally hyperbolic periodic arcs or simple closed curves.*

2. $\mathcal{R}$*, finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*

3. $\tilde{\mathcal{L}}(f)$ *can be decomposed into a disjoint union of finitely many compact invariant and transitive sets (called basic sets). Furthermore* $f_{|\tilde{\mathcal{L}}(f)}$ *is expansive.*

1 Billiards

Let B be an open bounded and connected subset of the plane whose boundary consists of a finite number of closed C^k-curves Γ_i, $i = 1, \cdots, m$.

The billiard map is a C^{k-1} diffeormorphism.

We assume that B is simple connected.

Non-elastic Billiards

ϕ_i: angle from the reflected vector to the inward normal $n(q_i)$.

The N-E billiard map is $P(r_0, \phi_0) = (r_1, \phi_1)$ where r_1 is obtained as in the usual billiard (moving along the direction determined by ϕ_0 beginning at the boundary point determined by r_0) and

$$-\pi/2 \leq \phi_1 = -\eta_1 + f(r_1, \eta_1) \leq \pi/2$$

where η_1 is the angle from the incidence vector at q_1 to the outward normal $-n(q_1)$ and
$f : [0, |\Gamma|] \times [-\pi/2, \pi/2] \to \mathbb{R}$ is a C^2 function.

A1. We assume that the perturbation depends only on the angle of incidence: $f = f(r, \eta) = f(\eta)$ for $-\pi/2 \leq \eta \leq \pi/2$, with $\eta \times f(\eta) \geq 0$.
Let us call $\lambda(\eta) = 1 - f'(\eta)$; $\lambda_i = 1 - f'(\eta_i)$.

In different works we have added some additional global conditions.

The following one is the main one for the numerical results (Arroyo, Markarian, Sanders):

A1b. We also assume that $f(0) = 0$ and that for a fixed constant $\lambda < 1, 0 \leq \lambda(\eta) < \lambda$.

A typical model for this case is $\lambda(\eta) = \lambda < 1$: there is uniform contraction, $f(\eta) = (1 - \lambda)\eta$ and the angle of reflection is $\phi = -\lambda\eta$ for $-\pi/2 \leq \eta \leq \pi/2$.

The trajectory moves approaching to the normal line in the reflection point: the absolute value of the angle (with the normal line) of reflection is smaller than or equal to the angle of incidence.

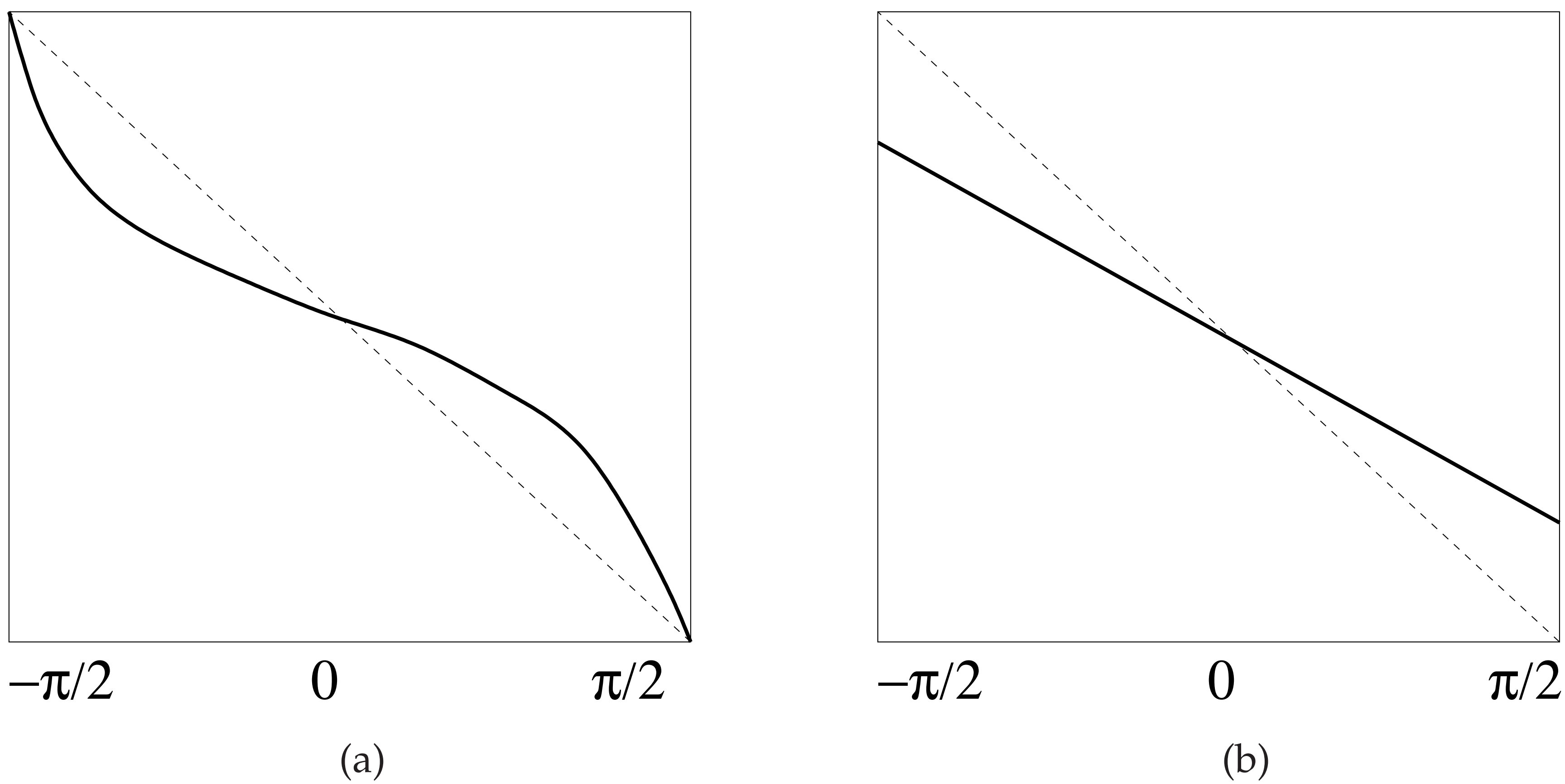


Figure 1: Graphics of $\phi = -\eta + f(\eta)$ for assumptions A1a and A1b.

The derivative $D_{x_0}T$ of the N-E billiard map satisfying Condition A1 at $x_0 = (r_0, \phi_0)$ is given by

$$-\begin{pmatrix} A & B \\ (K_1 A + K_0)\lambda_1 & (K_1 B + 1)\,\lambda_1 \end{pmatrix} \tag{1}$$

$$A = \frac{t_0 K_0 + \cos\phi_0}{\cos\eta_1} \quad ; \quad B = \frac{t_0}{\cos\eta_1}$$

This formula includes the angle of reflection and the angle η of incidence in the perturbed billiard.

If $f(r,\eta) \equiv 0$, then $\phi = -\eta$ and we have a elastic billiard map

If $f_\eta = f' = 1 \Longrightarrow \lambda = 0$, then the reflecting angle is constant, ϕ_0. The resulting one dimensional dynamical system has derivative

$$\frac{t_0 K_0 + \cos\phi_0}{-\cos\eta_1}$$

(its dynamical behavior depends on the curvature K and the distance between bouncing points) and is defined on the union of a finite number of arcs of finite length.

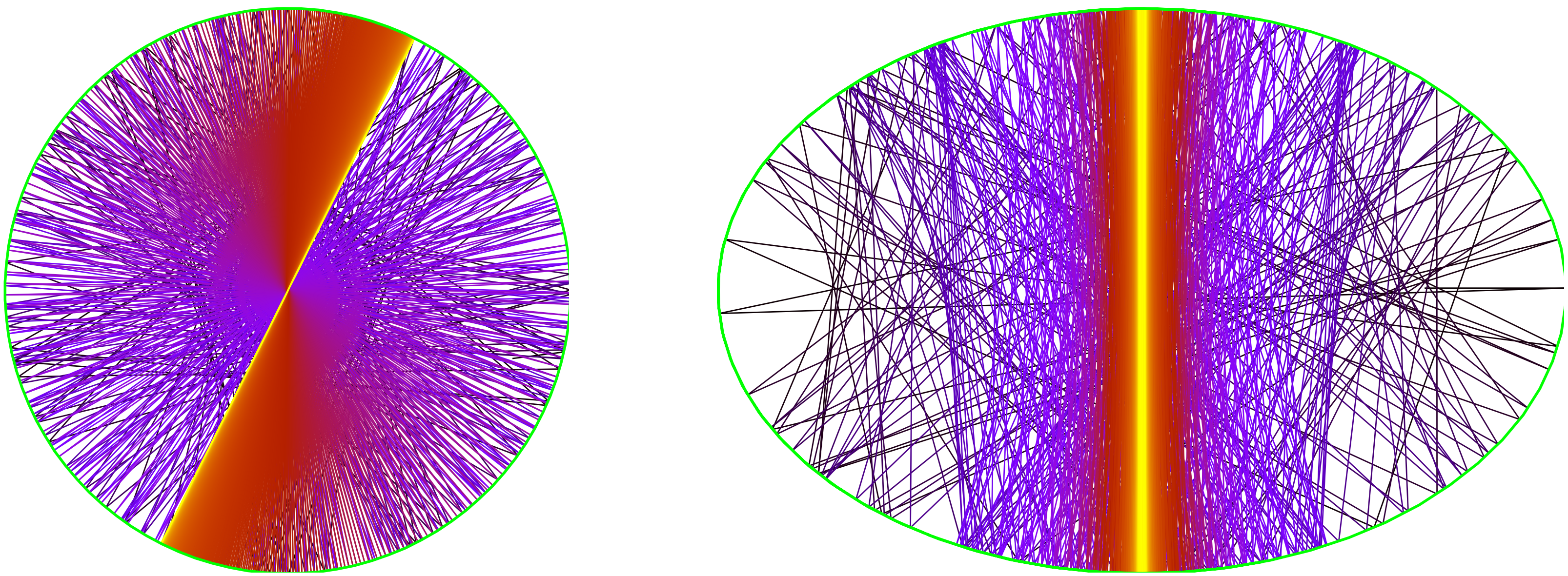
Extreme case: the particle reflects at the boundary along the normal line. We call it, *slap billiard map*.

Theorem 1. *The pinball billiard map associated to a billiard table satisfying Assumption* **A1b** *with non negative curvature (semidispersing walls) has a dominated splitting.*

This result includes billiards with cusps and polygonal billiards.

We have proved [MPS] results on pinball-billiards with focusing components of the boundary, curvature bounded away from zero ($-K > c > 0$), satisfying Assumptions **A1b**, or other technical conditions on the function f

Theorem 2. *Consider the pinball billiard map associated to a billiard table bounded by C^3 curves that are C^2 close to circle. If it satisfies Assumption* **A1b** *it has dominated splitting.*



(a) (b)

Figure 2: Single trajectories, $\lambda = 0.99$ in (a) circular table, (b) elliptical table with $a = 1.5$. Colours indicate the number of bounces, with lighter colours corresponding to later times, asymptotic convergence to period-2 orbits. Initial condition in (a) is a random one; in (b) was taken close to the unstable period-2 orbit along the major axis, from which it rapidly diverges.

Theorem 3. *The pinball billiard map associated to a billiard table with focusing components satisfying Wojtkowski conditions for a elastic billiard map being hyperbolic (non-vanishing Lyapunov exponents) has dominated splitting.*

Wojtkowski's condition $t_0 > d_0 + d_1$ where $d_i = -\cos\phi_i / K_i,\ i = 0, 1$[1]. It is equivalent to $\frac{d^2R}{dr^2} < 0$, where $R(r)$ is the curvature of the curve.

[1]Note that d_i is the length of the subsegment of $\overline{q_0 q_1}$ contained in the disk $D(q_i)$ tangent to Γ at q_i with radius $R_i/2 = -1/(2K_i)$ (*disks of semi-curvature*)

Additional conditions on the other components of the boundary are:
- dispersing comps. not adjacent to focusing comps. must be outside the disks of semi-curvature of all focusing components;

- disks of semi-curv. of diff. foc. comps.: disjoint;

- angle of intersection of smooth pieces of the boundary must be greater than π if both are focusing; not less than π if one is focusing, other dispersing; and bigger than $\pi/2$ if one is dispersing, other flat.

Cardioid satisfies curvature's condition at all its points and $\frac{d^2R}{dr^2} < c < 0$. Then the cardioid admits C^4 perturbations, maintaining the hyperbolicity.

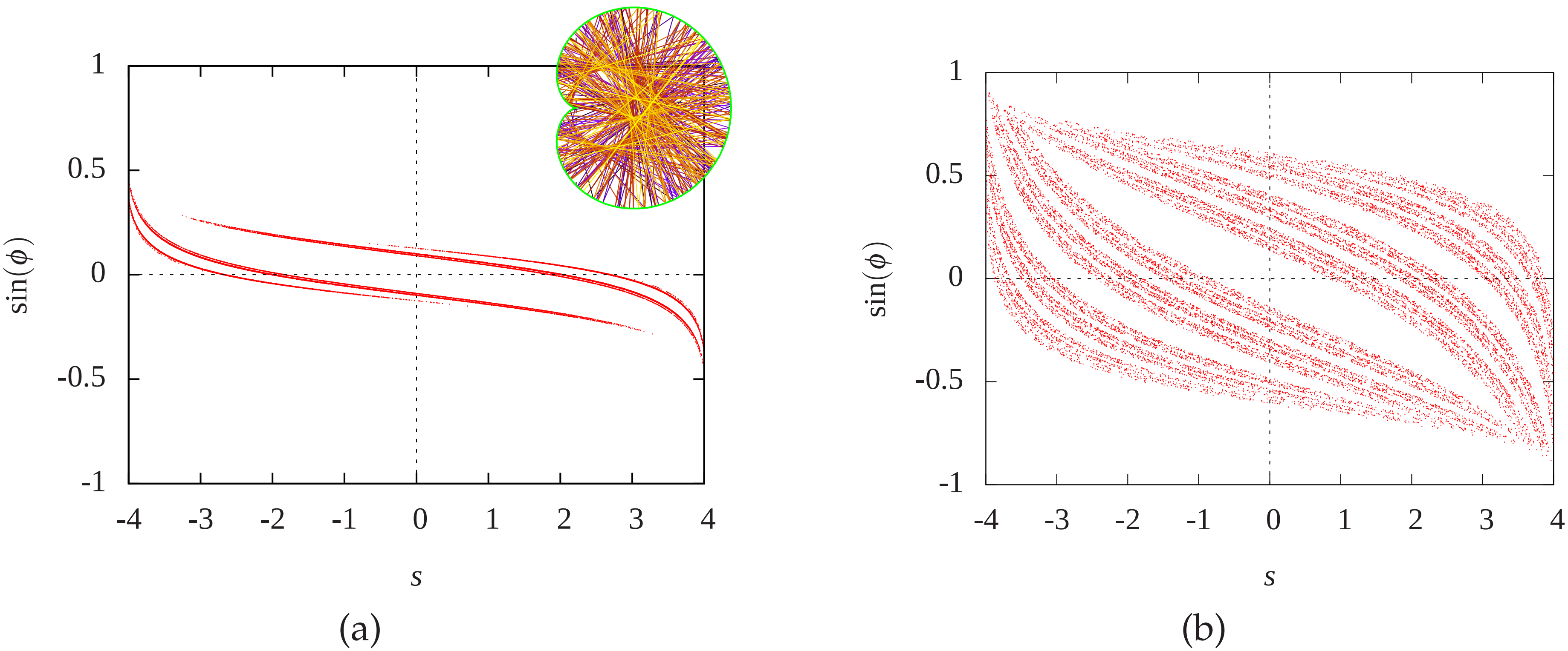
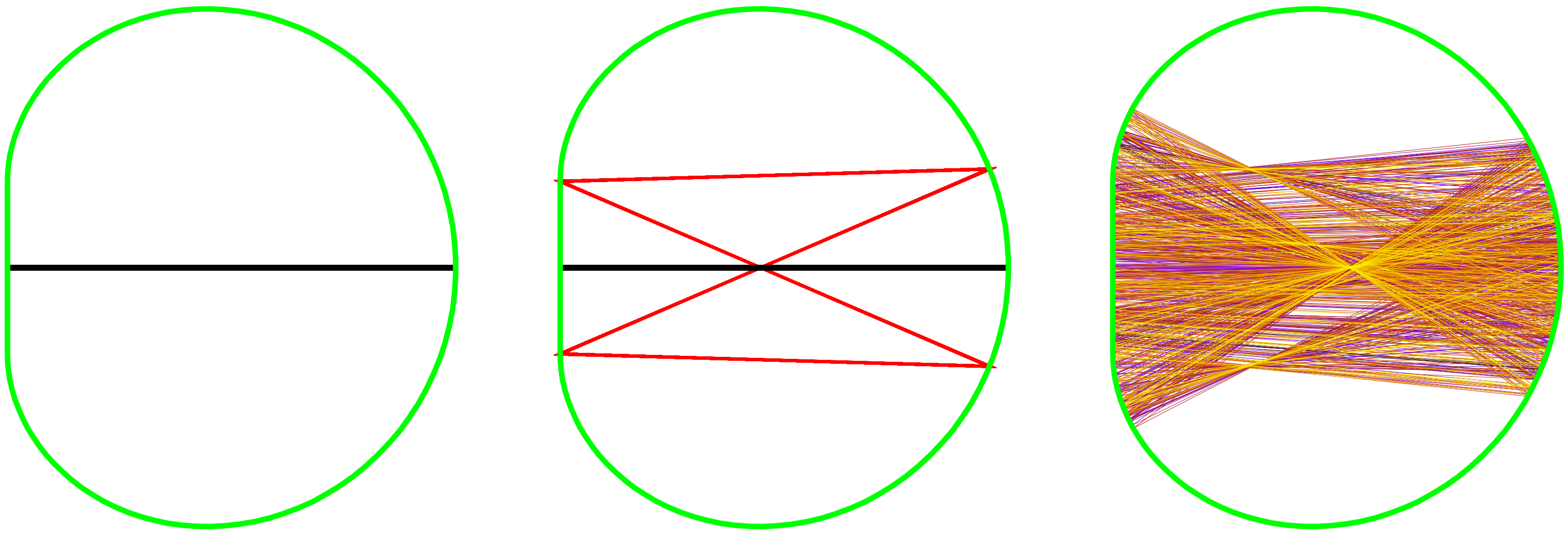
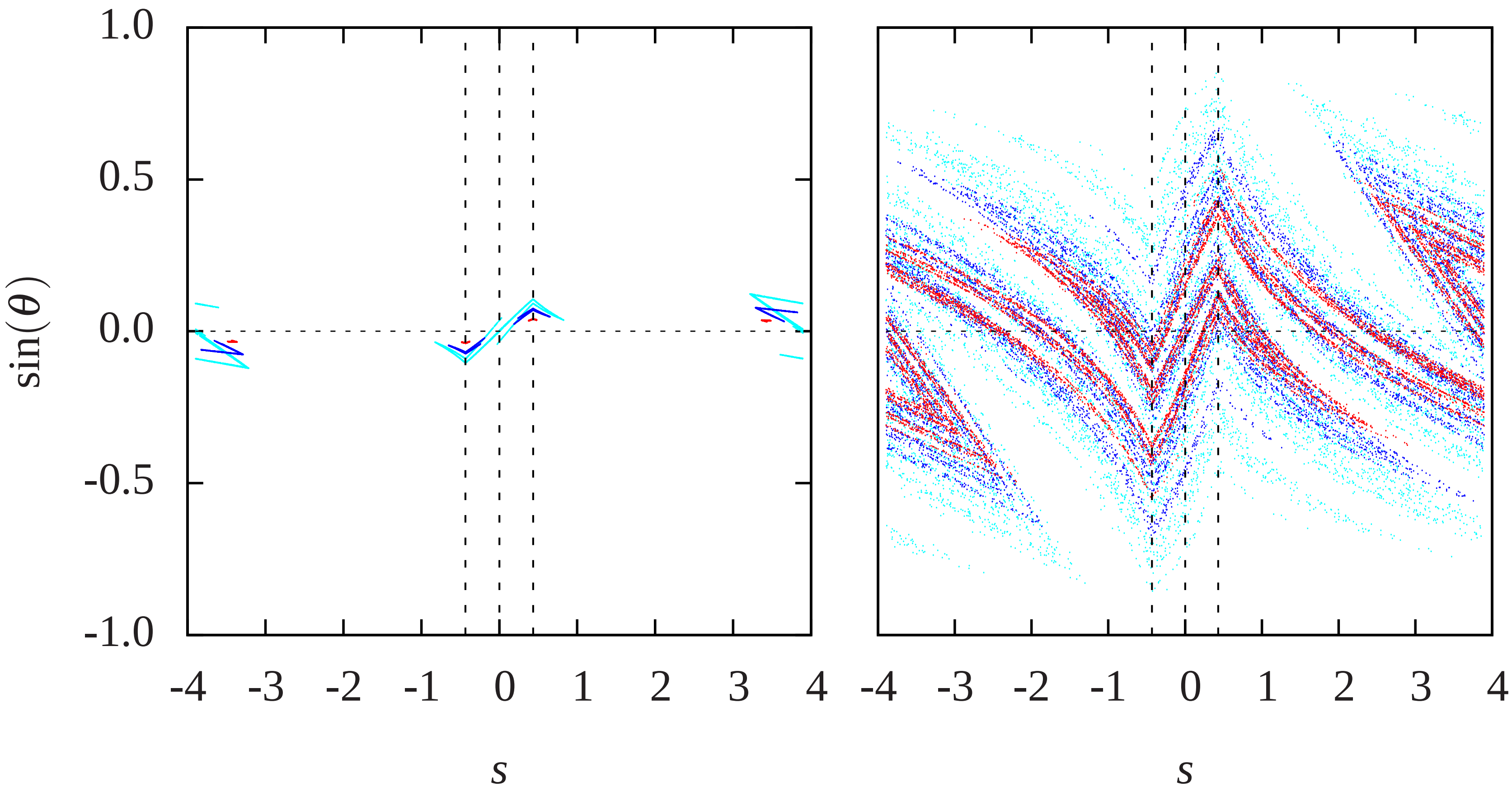


Figure 3: Cardioid $\rho(\theta) = 1 + \cos(\theta)$. Chaotic attractor, for (a) $\lambda = 0.3$ and (b) $\lambda = 0.8$. The inset of (a) shows the attractor in configuration space. Coordinates: arc length s and $\sin(\phi)$, where ϕ is the exit angle at each collision. The cusp of the cardioid is at $s = \pm 4$.



(a) $\lambda = 0.02$ (b) $\lambda = 0.072$ (c) $\lambda = 0.2$

Figure 4: Cuspless cardioid: Numerically-observed attractors in configuration space, with increasing λ. For $\lambda < \lambda_* \simeq 0.0712$: just a period-2 attractor. This periodic attractor coexists with a chaotic attractor for $\lambda \in [\lambda_*, \lambda_c]$, where $\lambda_c \simeq 0.093$. Period-2 attractor then becomes unstable, leaving just the chaotic attractor, which expands for increasing λ.



(a) $\lambda \in [0.1, 0.3]$ (b) $\lambda \in [0.7, 0.9]$

Figure 5: Cuspless cardioid: chaotic attractor; different colours (red, blue and cyan) indicate increasing order of λ. Vertical lines mark the centre of the vertical section and the two curvature discontinuities at $s = \pm\sqrt{3}/4$.

Now we recall a general method for establishing hyperbolic properties of dynamical systems [**?**, **?**].

Let M be a compact Riemannian manifold (perhaps, with boundary and corners) of dimension d, $M' \subset M$ an open and dense subset and $F\colon M' \to M$ a C^r (with $r \geq 1$) diffeomorphism of M' onto $F(M')$. M' is the union of a finite number of open connected sets M_i^+. Note that all the iterations of F are defined on the set

$$\tilde{M} = \cap_{n=-\infty}^{\infty} F^n(M').$$

Let m be the Lebesgue measure on M. We will assume that $\tilde{M}$ has full measure: $m(M) = m(\tilde{M})$.

We recall that a quadratic form Q in $\mathbb{R}^d$ is a function $Q\colon \mathbb{R}^d \to \mathbb{R}$ such that $Q(u) = Q_2(u,u)$, where Q_2 is a bilinear symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$.

Equivalently, $Q\colon \mathbb{R}^d \to \mathbb{R}$ is a quadratic form if there is a symmetric matrix A such that $Q(u) = u^T A u$ for $u \in \mathbb{R}^d$ (here u^T means transposition of a column-vector u).

A *quadratic form* Q on M is a function $Q : \mathcal{T}M \to \mathbb{R}$ such that its restriction Q_x to $\mathcal{T}_x M$ at m-almost every point $x \in M$ is a quadratic form in the usual sense.

We say that a quadratic form Q is *nondegenerate* at x if for every nonzero vector $u \in \mathcal{T}_x M$, there exists a $v \in \mathcal{T}_x M$ such that $Q_2(v, u) \neq 0$ (equivalently, $\det A \neq 0$ for the corresponding symmetric matrix A).

We say that Q is *positive* (nonnegative) if at every point x the form Q_x is positive definite (positive semidefinite); i.e. $Q_x(u) > 0$ (respectively, $Q_x(u) \geq 0$) for all $0 \neq u \in \mathcal{T}_x M$.

Let be Q a nondegenerate quadratic form defined on $\mathcal{T}M$ with positive index of inertia equal to p and negative index of inertia equal to q, $p+q = d, p \geq 1, q \geq 1$, for every $x \in M$.

We assume that Q is continuous on each M_i^+ and denote by

$$C_{\pm}(x) = \{v \in \mathcal{T}_x M : \pm Q_x(v) > 0\} \cup \{0\}$$

the open cones of, respectively, positive and negative vectors (with the zero vector included), and by $C_0(x)$ their common bound., $C_0(x) = \{v \in \mathcal{T}_x M : Q_x(v) = 0\}$.

$D_xT : \mathcal{T}_xM \to \mathcal{T}_{Tx}M$ is

1. Q-separated if $D_xTC_+(x) \subset C_+(Tx)$,
2. strictly Q-separated if $D_xT(C_+(x) \cup C_0(x)) \subset C_+(Tx)$,
3. Q-monotone if $Q_{Tx}(D_xTu) \geq Q_x(u)$ every $u \in \mathcal{T}_xM$,
4. strictly Q-monotone if $Q_{Tx}(D_xTu) > Q_x(u)$ for every $u \in \mathcal{T}_xM, u \neq 0$.

3. $\Longrightarrow$ 1. 4 $\Longrightarrow$ 2.

In Wojtkowski: Monotonicity, $\mathcal{J}$-algebra of Potapov and Lyapunov exponents, Proceed. of Symposia in Pure Maths., **69**, AMS (2001), following some remarkable works by V. P. Potapov, it is proved that

1. If DT is Q-separated then the set of positive numbers r such that $\frac{1}{r}DT$ is Q-monotone is a closed interval possibly degenerating to a point.
   $r \in [r_-, r_+],\ r_- > 0$, with

$$r_-^2(x) = \sup_{u \in C_-(x)} \frac{Q_{Tx}(D_x T u)}{Q_x u}, \tag{2}$$

$$r_+^2(x) = \inf_{u \in C_+(x)} \frac{Q_{Tx}(D_x T u)}{Q_x u}\ . \tag{3}$$

2. If DT is strictly Q-separated then the set of positive numbers r such that $\frac{1}{r}DT$ is strictly Q-monotone is an open interval: $(r_-, r_+), r_- > 0$.

Definition 1. *$DT : TM \to TM$ is* eventually uniformly strictly Q-separated *(euss) at x if it is Q-separated in every point $T^i(x)$, $i \in \mathbb{Z}$ of the orbit of x, and there exist constants $m \geq 1$ and $0 < d < 1$ (not depending on x and n) such that for each $n \geq 0$*

$$\#\{i : D_{F^{n+i}x}FC_+(T^{n+i}x) \text{ is not strictly contained in } C_+(F^{n+i+1}x)\} \leq m \ \text{ and}$$

$$\#\left\{j : 0 \leq j \leq m,\ \frac{r_-(T^{n+j}x)}{r_+(T^{n+j}x)} \leq d\right\} > \ 0\,. \tag{4}$$

Definition 2. *The diffeomorphism F is euss in an invariant set N if DF is euss at each point $x \in N$.*

Proposition 4. *If the diffeomorphism F is euss in an invariant set N then N has a dominated splitting.*

Proof. Is similar to the proof of Proposition 4.1 in [**?**] (see also Proof of Theorem 1 in [**?**]).
Conditions for F being euss are automatically satisfied in the original proof because it is assumed that F acts on a compact manifold. . If F preserves a probability measure, the exponential contraction of the diameter of the manifold of (positive) linear subspaces contained in C_+ is obtained by standard methods (using the Birkhoff Ergodic Theorem).

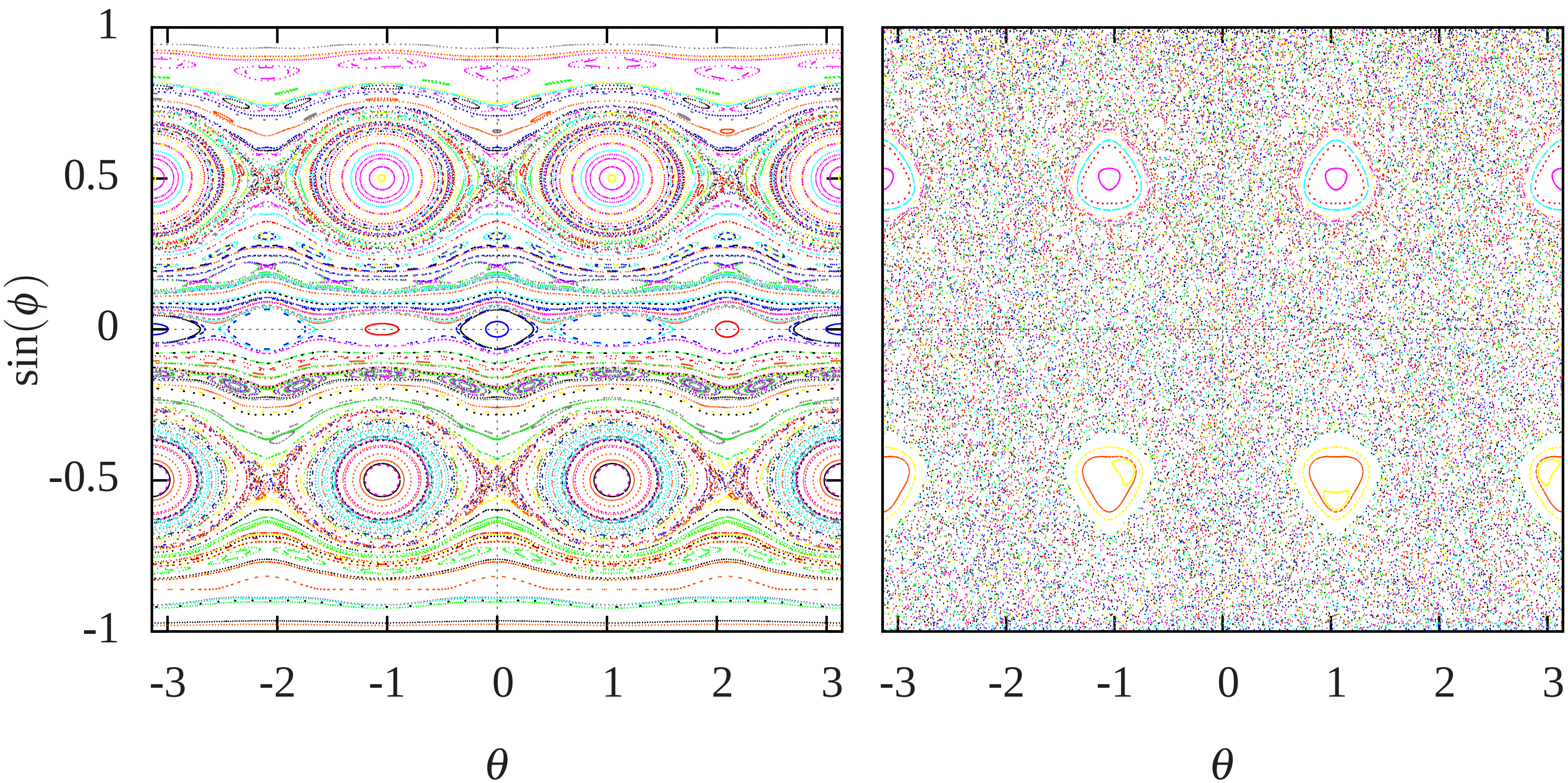
But we are not using invariant measures. Then ...

Convex billiards

If B is strictly convex, sufficiently smooth boundary with curvature K ($0 < a < K < b$), the phase space is compact.

There exists positive measure set N in the billiard phase space M that is foliated by invariant curves. The set N accumulates on the horizontal boundary of M. (Lazutkin)

All trajectories starting in the set N have caustics, which are convex curves lying inside B.



(a) $\alpha = 0.02$ (b) $\alpha = 0.08$

Figure 6: Three-pointed egg, $\rho(\theta) = 1 + \alpha\cos(3\theta)$, Hamiltonian case $\lambda = 1$, Different colours indicate trajectories from different initial conditions. For $\alpha > 1/10$, table becomes non-convex

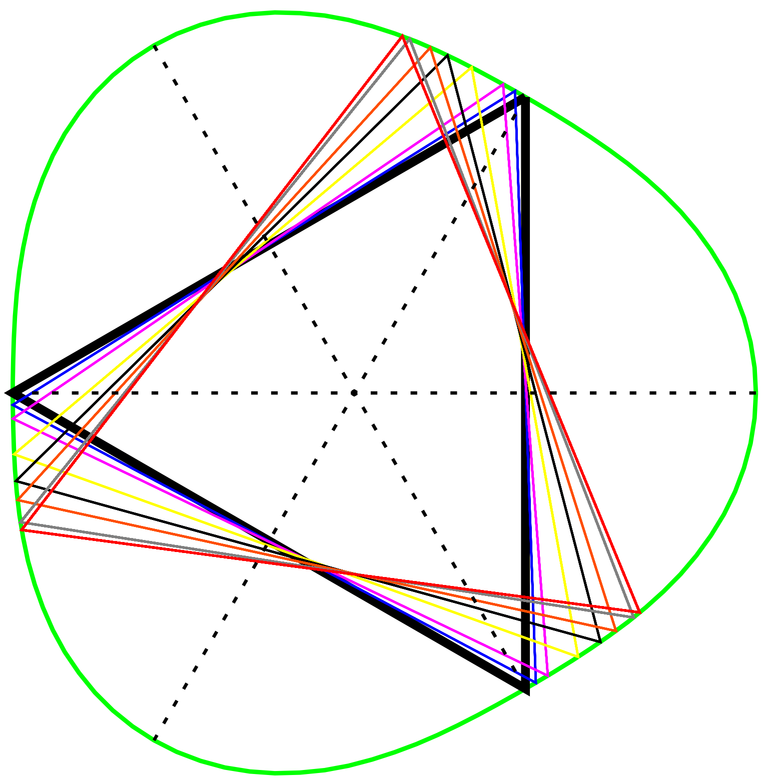
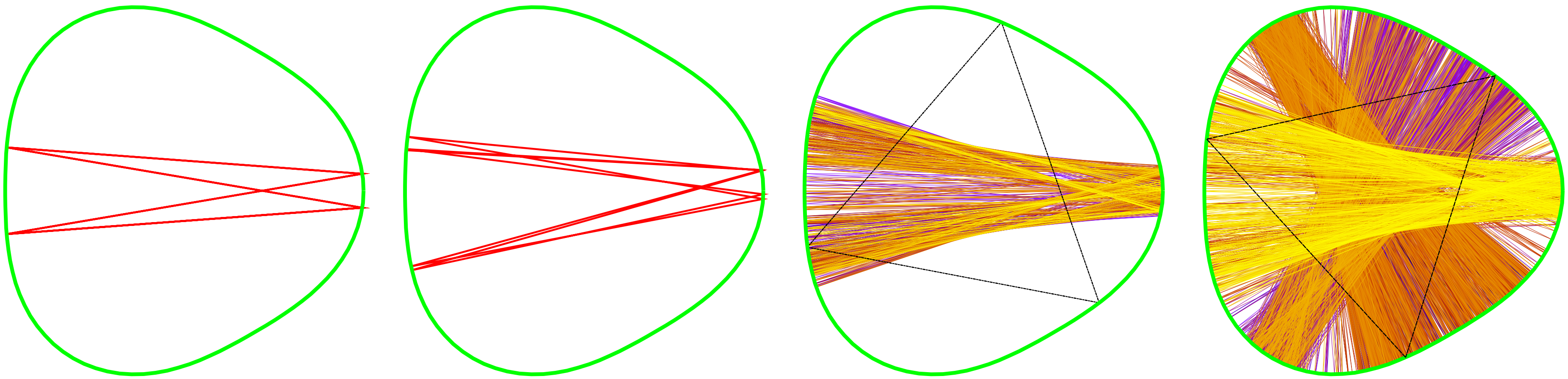


Figure 7: Change in position of one of the attracting period-3 orbits as λ is varied; shape parameter $\alpha = 0.08$. For $\lambda = 1$ (thick black line) the orbit is elliptic; for $\lambda < 1$ it is attracting. The values of λ shown are, in an anti-clockwise direction, $\lambda = 1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.45, 0.41, 0.40$. The latter value is close to the numerically-determined limit of existence of the period-3 orbits, which is $\lambda \simeq 0.39$.



(a) $\lambda = 0.1$ (b) $\lambda = 0.39$ (c) $\lambda = 0.43$ (d) $\lambda = 0.45$

Figure 8: Three-pointed egg: Attractors in configuration space, $\alpha = 0.08$: (a) period-4 orbit which persists from the stable period-4 orbit of the slap map ($\lambda = 0$); (b) period-8, after undergoing a single period-doubling bifurcation; (c): localized chaotic attractor, after the accumulation of period doublings; (d):trajectories tend to remain for a long time in each part of the attractor previously localised, before jumping to a different part, as shown by the colours in the figure. In each of (c) and (d), coexisting period-3 orbit is shown in black.

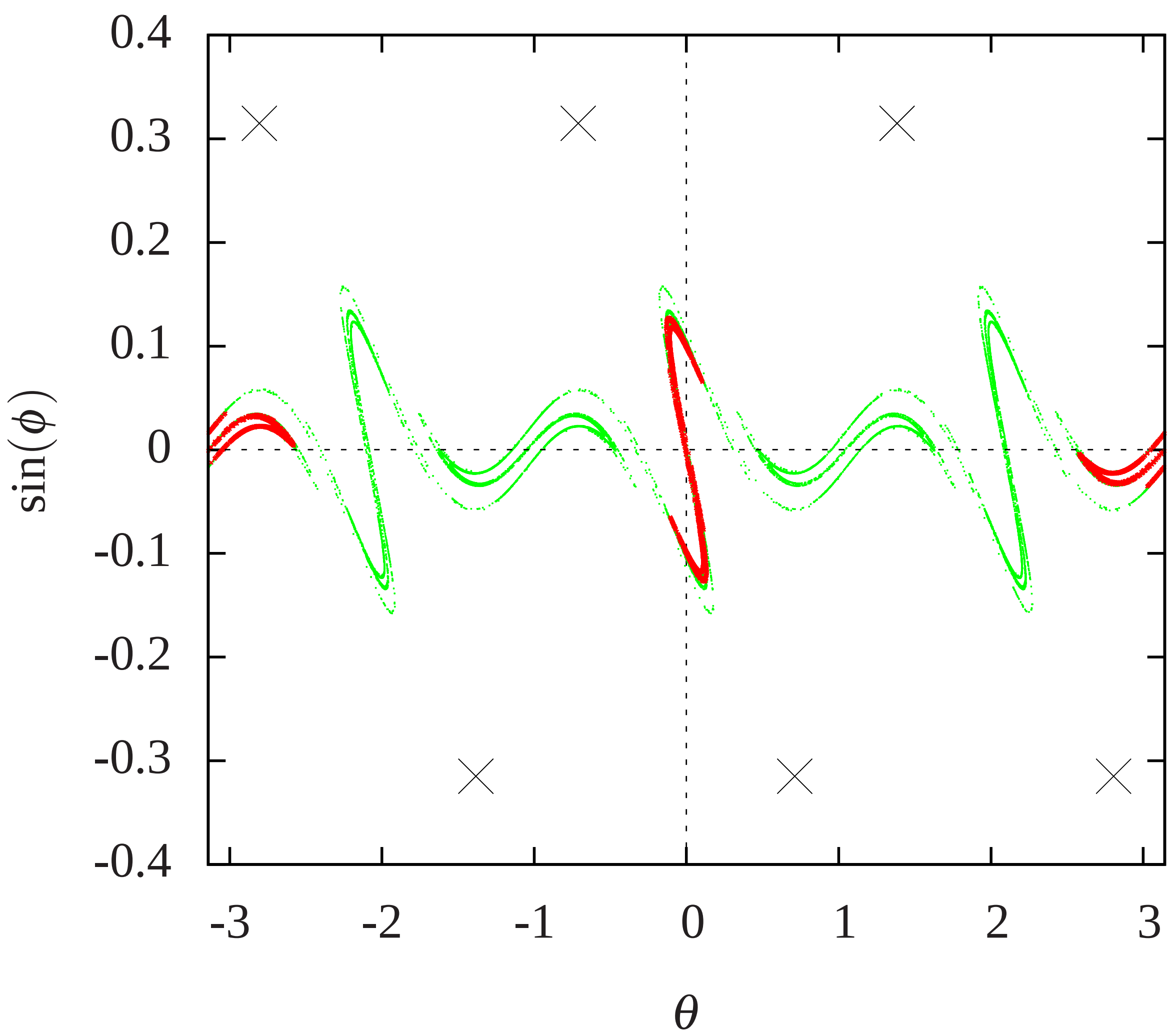
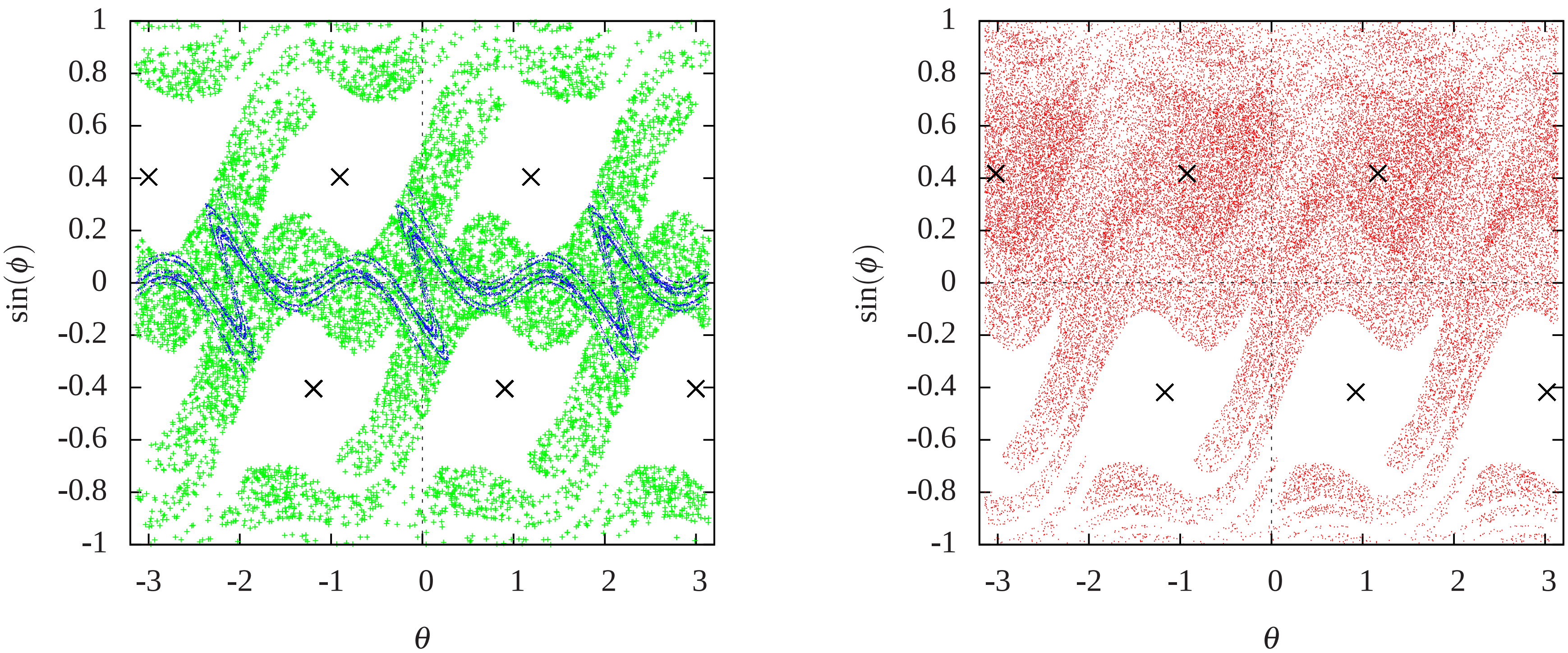


Figure 9: Attractors in phase space for the three-pointed egg with $\alpha = 0.08$ and $\lambda = 0.43$ (red), $\lambda = 0.45$ (green). The coexisting period-3 attractors are also shown (black $\times$).



(a) $\lambda = 0.66$ (b) $\lambda = 0.7$

Figure 10: Three-pointed egg: basins of attraction, $\alpha = 0.08$. (a) chaotic attractor (blue points), its basin of attraction (green); period-3 attracting orbits (black), their basins of attraction in white, are shown. (b) chaotic attractor has disappeared In red dots and blank, basins of each of period-3 orbits. The region that in (a) was occupied by the basin of the chaotic attractor is now a region where the basins of the two periodic orbits intermingle.

Lazutkin: In small neighbourhood of the stationary curves of billiard map T: a family of invariant closed curves. T is topologically equivalent to rotation (each invariant curve having its own small angle).

Rotation numbers can not be well approximated by rational numbers.

Family of caustics is not continuous: may not appear around some rational rotation numbers.

KAM theorem is the main instrument in its proof:

Let f be a volume preserving diff of class $C^r, r \geq 4$ of a surface M. x non degenerate elliptic fixed point, then given $\epsilon > 0 \;\; \exists$ arbitrary small neighborhood U of x and $U_0 \subset U$:
a) U_0 is a union of f-invariant simple closed curves of class C^{r-1} containing x in their interior;
b) the restriction of f to each of these curves is topologically equivalent to an irrational rotation;
c) $\mu(U \setminus U_0) \leq \epsilon\mu(U)$.

r = 4, Rûsmann (1970).

r = 3, Herman, Asterisque 103-104 (1983) and 144 (1986), with the loss of c)

There can be other invariant curves.

ϕ angle of the trajectory vector with the oriented tangent to the curve with radius of curvature R

$$x(\phi) = \int_0^\phi R(\beta)\cos\beta d\beta, \quad y(\phi) = \int_0^\phi R(\beta)\sin\beta d\beta.$$

If $R(\phi) = a + b\cos n\phi$, the billiard map has an invariant curve with irrational rotation on the line of constant angle α such that

$$n\tan\alpha = \tan n\alpha \ \text{ for } \ n \geq 4,\ a > b.$$

If $n = 4$, $\alpha \approx 66^o$.

Non elastic billiard maps: composition of a classical billiard followed by a change at the reflection angle. Let γ be a C^2 rotational invariant curve of T, given $\alpha = g(\varphi)$. The non elastic billiard map P is

$$P(\varphi_0, \alpha_0) = (\varphi_1, \alpha_1 - h(\varphi_1 - g(\alpha_1))$$

where $(\varphi_1, \alpha_1) = T(\varphi_0, \alpha_0)$ and $h : I \mapsto \mathbb{R}$ is a C^2 function, $0 \in I$, closed interval.

Compact strip: A compact subset of $[0, 2\pi) \times (0, \pi)$ with non-empty interior and whose boundaries are two distinct rotational curves (not necessarily invariant nor graphs).

Since M is compact, it is much more simple to prove the existence of dominated splitting.

Let $u, v : M \mapsto \mathcal{T}M$ be two vector fields such that for each $x \in M$, $u(x) = u_x$ and $v(x) = v_x$ are two linearly independent vectors in the tangent space $\mathcal{T}_xM$. Continuous vector fields $\Rightarrow$ continuous cone field.

$[Df_x]_U$ is the matrix representation of the derivative at x, with the choice of $\{u_x, v_x\}$ and $\{u_{fx}, v_{fx}\}$ as bases of $\mathcal{T}_xM$ and $\mathcal{T}_{fx}M$ respectively.

Lemma 5. *Let Λ be a compact f-invariant subset of M. If the entries of $[Df_x]_U$ are strictly positive for every $x \in \Lambda$ then Λ has a dominated splitting.*

A (non homotopic to a point) continuous closed curve γ on the cylinder $[0, 2\pi) \times (0, \pi)$ is called a rotational curve.

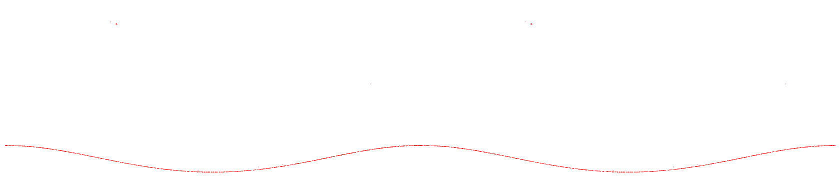
As T preserves area, two distinct invariant rotational curves do not intersect. This, together with the reversibility of T and the compactness of γ, imply that either $g(\varphi) \equiv \frac{\pi}{2}$ or there exist constants b and B such that

$$0 < b \leq g(\varphi) \leq B < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < B \leq g(\varphi) \leq b < \pi.$$

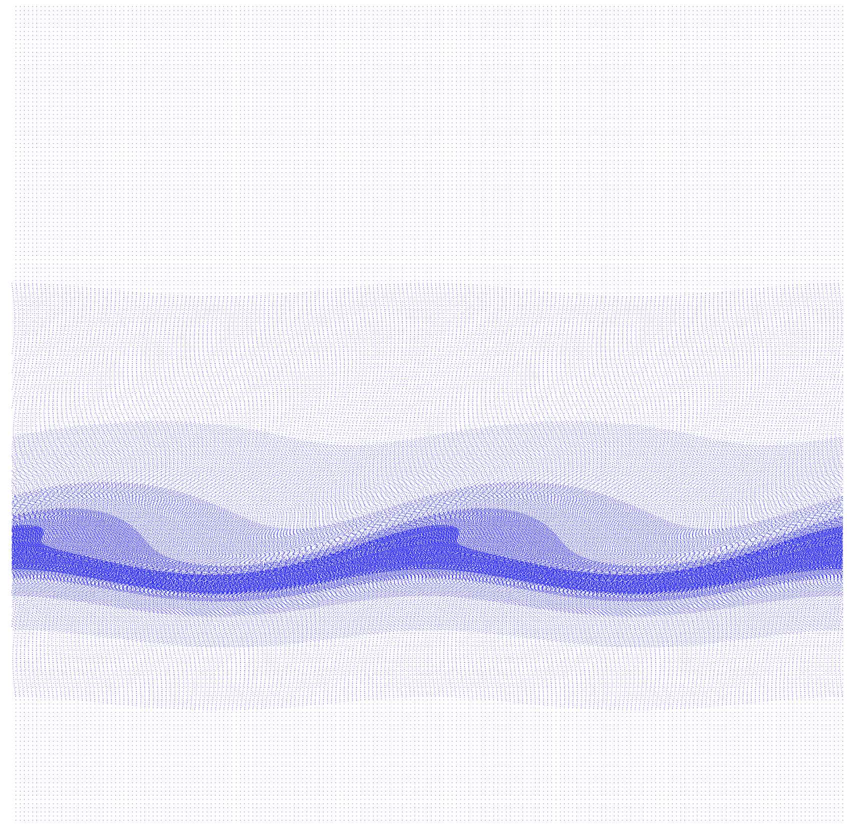
Proposition 6. *Given a classical billiard map T on an oval, with a C^2 invariant rotational curve $\gamma = \{(\varphi, g(\varphi))\}$, there exist a closed interval I, containing 0 in its interior, a C^2-function*

$$h : I \mapsto \mathbb{R}, \quad h(0) = 0, \quad 0 < \mu \leq h'(t) \leq \lambda < 1$$

and a compact strip S such that the non elastic billiard P defined by T, g and h is a C^2-diffeomorphism from S onto $P(S)$ and $L(P) \cap S$ contains γ and has a dominated splitting. Moreover, the non elastic dynamics on γ is determined by its rotation number with respect to T.

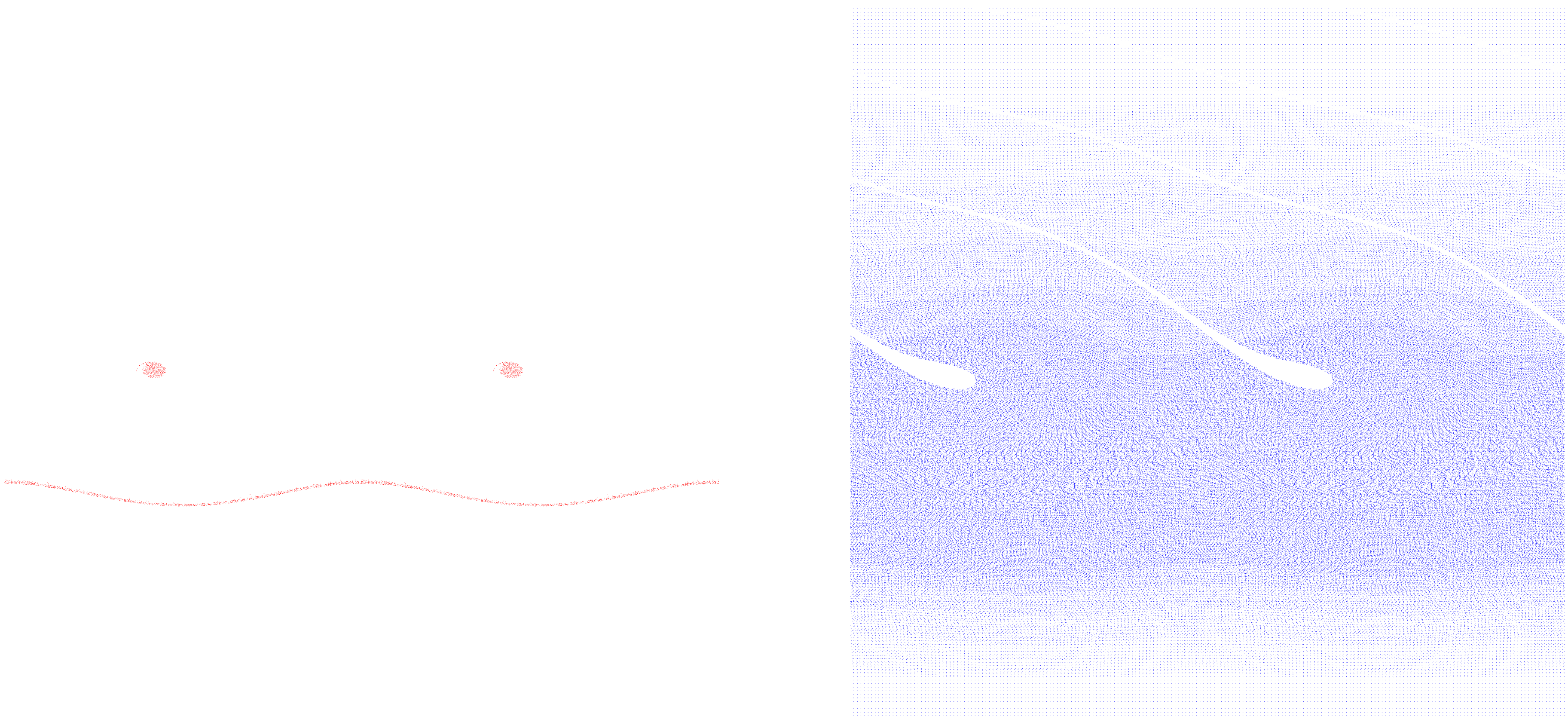


(a) *Attractor* γ_0



(b) Basin of attraction

Figure 11: Ellipse, eccentricity $e = 0.35$. Contraction $\mu = 0.5$. The simulation indicates that γ_0 is the unique attractor .

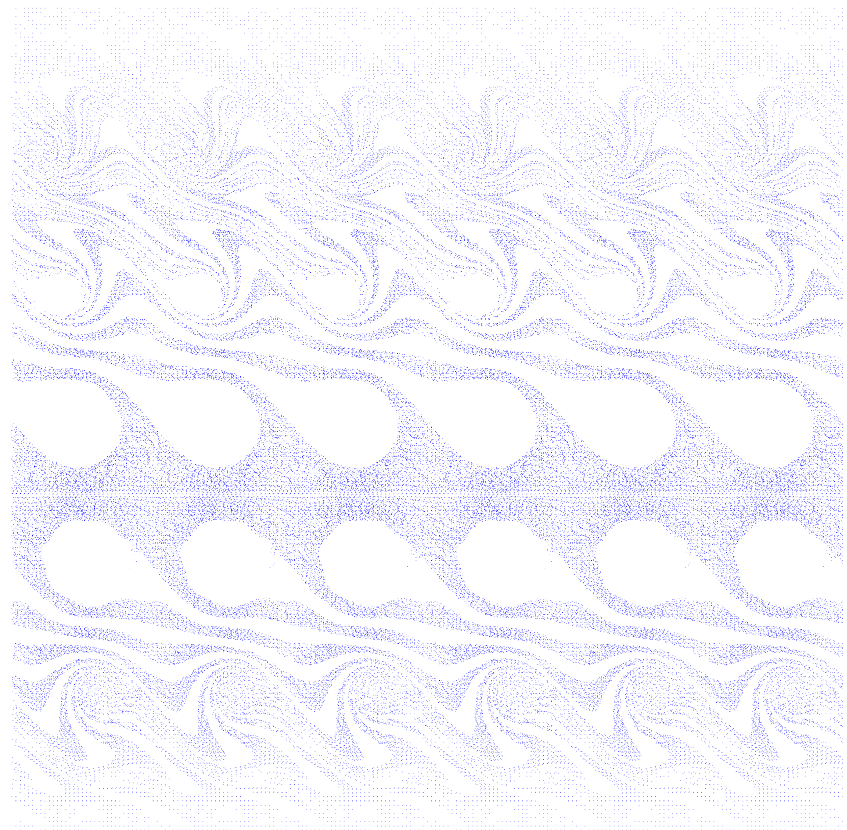


(a) Attractors: γ_0 and period-2 orbit

(b) Basin of attraction

Figure 12: Ellipse, eccentricity $e = 0.35$. Contraction $\mu = 0.2$. The simulation indicates that there is a period-2 attractor; γ_0 is not the unique attractor .

(a) Attractors: γ_0 and periodic orbits;



(b) Basin of attraction

Figure 13: Nonintegrable billiard; $n = 6$. Contraction $\mu = 0.1$. The simulation indicates that there are periodic attractors; γ_0 ($\alpha = \arctan\sqrt{7 + 4\sqrt{21}/3} \simeq 0.41\pi$) is not the unique attractor .