How logarithm laws may fail in a mixing system: an example with a reparametrization of a translation on the torus

Pietro Peterlongo (joint work with Stefano Galatolo)
PhD student at Scuola Normale, Pisa and Ecole Normale, Paris (Laboratoire de Météorologie Dynamique)

Hyperbolic Dynamical Systems in the Sciences
INdAM, Corinaldo, June 4, 2010

Plan of the talk

I. Classical results for logarithm laws

A. Logarithm laws for hitting time (fast decay)

B. Failure in the case of a strangely Liouvillean torus translation

C. Mixing reparametrization (on $\mathbb{T}^3$) and subpolynomial decay of correlations.

Galatolo, P., Long hitting time, slow decay of correlations and arithmetical properties, Disc Cont Dyn Syst A, 27 (2010), also on arXiv

Classical result 1/2 (Erdos-Rényi 1970)

Theorem

Let R_n be the longest run of heads after n coin tossing, then with probability one

$$\lim_{n\to\infty} \frac{R_n}{\log_{\frac{1}{p}}(n)} = 1$$

where $p =$probability of head> 0.

- 1=Head, 0=Tail
- Coin→0101110101....
- $R_2 = 1, \ \dots , R_{10} = 3$

If $p = \frac{1}{2}$ then for almost every coin tossing

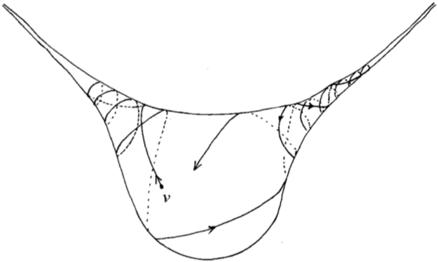
$$\lim_{n\to\infty} \frac{R_n}{\log_2(n)} = 1$$

Classical result 2/2 (Sullivan 1982)

Theorem

If $Y = H^{d+1}/G$, with G discrete subgroup of isometries s.t. Y is not compact and has finite volume. Let T^1Y be the unit tangent bundle and $\pi : T^1Y \to Y$ the canonical projection. Let ϕ^t be the geodesic flow and μ the Liouville measure. Then $\forall p \in Y$ and μ a.e. $v \in T^1Y$

$$\limsup_{t\to\infty} \frac{dist(p, \pi(\phi^t v))}{\log t} = \frac{1}{d}$$



Waiting (hitting) time

Let T be a transformation on X with invariant measure μ.

Definition

$\tau_A(x)$ is the time x first enters in a set A

$$\tau_A(x) := \min\{n > 0 : T^n(x) \in A\}$$

- *alternative definition of ergodicity:* for every positive measure set A, waiting time is almost everywhere finite.

Example

For a rotation on $\mathbb{S}^1$, i.e. a translation of $\alpha \in (0,1)$ of $\mathbb{R}/\mathbb{Z}$, consider $A = B_r(0)$.
The set of possible waiting times for $r > 0$ is precisely the set of denominators of convergents q_n of the (regular) continued fraction expansion of α.

Hitting (waiting) time indicator

Let X be a metric space.

Take a target point x_0. Let $A = B_r(x_0)$.
We wish to study $\tau_r(x) := \tau_A(x)$ as $r \to 0$.

Example

For a rotation on $\mathbb{S}^1$ take 0 as target point; the radii at which $\tau_r(0)$ is discontinous are exactly $\|q_n\alpha\|$.

hitting time indicator:

$$\overline{R}(x, x_0) = \limsup_{r \to 0} \frac{\log \tau_r(x, x_0)}{-\log r}$$

$$\underline{R}(x, x_0) = \liminf_{r \to 0} \frac{\log \tau_r(x, x_0)}{-\log r}$$

Logarithm laws for hitting time - local dimension

Kac's theorem says that $\mathbb{E}(\tau_A | x \in A) = \frac{1}{\mu(A)}$, so logarithm laws for hitting time should be of the form

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log \mu(B_r)} = 1$$

or, defining local dimension at x_0 as $d_\mu = \lim_{r \to 0} \frac{\log \mu(B_r(x_0))}{\log r}$, in this form

$$\lim_{r \to 0} \frac{\log \tau_r(x)}{-\log r} = d_\mu$$

Example

Take the distance between sequences of 0s and 1s which is the sum of all $\frac{1}{2^i}$ for each index i where the sequences differ.
The local dimension of probability measure of Bernoulli shift (coin tossing) is 1.

Dictionary: hitting time $\Longleftrightarrow$ distance

Let d be the distance in X and $d_n(x, x_0) = \min_{i \leq n} d(T^i(x), x_0)$.

$$\frac{1}{\underline{R}(x, x_0)} = \limsup_{n \to \infty} \frac{-\log d_n(x, x_0)}{\log n} \quad (1)$$

$$= \sup\{\beta : \liminf_n n^\beta d_n(x, x_0) = 0\} \quad (2)$$

$$= \limsup_n \frac{-\log d(T^n(x), x_0)}{\log n} \quad (3)$$

Boshernitzan, Chaika *(preprint 09)*

If an Interval Exchange Transformation is ergodic (relative to the Lebesgue measure λ), then the equality

$$\liminf_{n \to \infty} n|T^n(x) - y| = 0$$

holds for $\lambda \times \lambda$-a.a. x, y.

Logarithm law for fast mixing systems - Thm A

A system has superpolynomial decay of correlations (for Lipschitz observables) if

$$\left| \int \phi \circ T^n \psi d\mu - \int \phi d\mu \int \psi d\mu \right| \leq \|\phi\| \cdot \|\psi\| \, \Phi(n)$$

holds for all ϕ, ψ Lipschitz observables, with Φ having superpolynomial decay (i.e. $\lim n^\alpha \Phi(n) = 0, \forall \alpha > 0$)

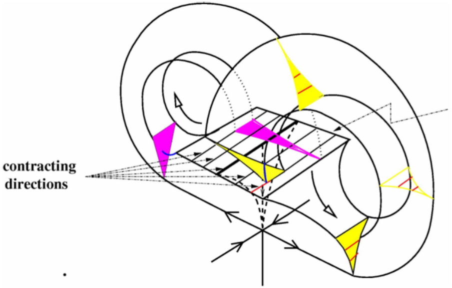
Theorem A *(Galatolo 07)*

If (X, T, μ) has superpolynomial decay of correlations and $d_\mu(x_0)$ exists then

$$\underline{R}(x, x_0) = \overline{R}(x, x_0) = d_\mu(x_0)$$

for μ-almost every x.

Applications - geometric Lorenz flow



Theorem *(Galatolo, Pacifico 10)*

The geometric Lorenz flow satisfies a logarithm law. For each x_0 where $d_\mu(x_0)$ is defined, we have for μ a. e. x:

$$\lim_{r \to 0} \frac{\tau_r(x, x_0)}{-\log r} = d_\mu(x_0) - 1$$

Failure of logarithm law

Problem

When a logarithm law can fail?
What needs to go wrong for hitting time indicators $(\underline{R}, \overline{R})$ to be different from local dimension?

Failure of logarithm law

Problem

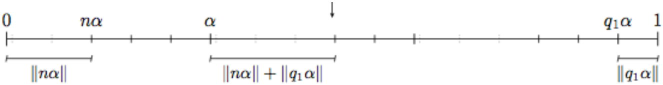
When a logarithm law can fail?
What needs to go wrong for hitting time indicators $(\underline{R}, \overline{R})$ to be different from local dimension?

For the case of irrational rotation on the circle we will see that $\overline{R}$ is the type of the rotation number, which can be any number bigger than 1 (even infinite, Liouville case).
Still, $\underline{R} = d = 1$.

Theorem B *(Galatolo, P. 10)*

If $T_{(\alpha,\alpha')}$ is a translation of the two torus by a vector $(\alpha, \alpha') \in Y_\gamma$, then for almost every $x \in \mathbb{T}^2$

$$\underline{R}(x, x_0) \geq \gamma$$



Three distances' Theorem *(informal dynamic version)*

For a rotation on $\mathbb{S}^1$, the orbit of 0 partitions the circle in intervals which have (at most) three lengths. The big interval is the sum of the small and the median.
A new iterate cuts out a small interval from a big interval, until you run out of big intervals.
Sometimes (at $t = q_n$) , this process produces a new small interval (of length $\|q_n\alpha\|$).

1. $\|q_{n+1}\alpha\| = \|q_{n-1}\alpha\| - a_n\|q_n\alpha\|$
2. $q_{n+1} = q_{n-1} + a_n q_n$
3. $\frac{1}{q_{n+1}} > \|q_n\alpha\| > \frac{1}{q_n+q_{n+1}} \left(> \frac{1}{2}\frac{1}{q_{n+1}}\right)$

Type of an irrational - Kim and Seo's result

The type β of an irrational number α is defined in one of the two following equivalent ways:

$$\gamma(\alpha) := \sup\{\beta : liminf_{n\to\infty} n^{\beta}\|n\alpha\|\} = \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}$$

The set of numbers of type γ has Hausdorff dimension $\frac{2}{\gamma+1}$.
The set of number of infinite type – Liouville numbers – is uncountable and dense.

Ex. A quantitative recurrence result: $\liminf_{r\to 0} \frac{\log \tau_r(x,x)}{-\log r} = \frac{1}{\gamma}$

Theorem *(Kim, Seo 03)*

If $(\mathbb{S}^1, T_\alpha, \lambda)$ is a rotation of the circle, $x_0 \in \mathbb{S}^1$ and γ is the type of α, then for almost every x

$$\overline{R}(x, x_0) = \gamma, \qquad \underline{R}(x, x_0) = 1$$

a strangely Liouvillean vector

Take $\gamma > 1$ and let $Y_\gamma \subset \mathbf{R}^2$ be the class of couples of irrationals (α, α') given by the following conditions on their convergents to be satisfied eventually:

$$q'_n \geq q_n^\gamma;$$

$$q_{n+1} \geq {q'_n}^\gamma.$$

We note that each Y_γ is uncountable and dense in $[0,1] \times [0,1]$ and each irrational of the couple is of type at least γ^2. The set $Y_\infty = \bigcap_\gamma Y_\gamma$ is also uncountable and dense in unit square and both coordinates of the couple are Liouville numbers .

Why strange? Each of the coordinate is very well approximate by a rational number. It takes a long time to distinguish the corresponding circle transformation from a periodic rotation. But they are 'periodic' at different times so the compound effect is not so well approximable by a rational vector.

sketch of proof of Thm B (a Borel-Cantelli argument)

- continous limit is preserved along the sequence $r_i = e^{-i}$
- consider the sequence of subsets of the two torus:

$$\overline{A}_i := \{\tau_{r_i}^{(\alpha,\alpha')}(x) < (2r_i)^{-\beta}\} = \{\frac{\log \tau_{r_i}^{(\alpha,\alpha')}(x)}{-\log r_i} + \frac{\beta \log 2}{-\log r_i} < \beta\}$$

  it is sufficent to prove that they are summable

$$\mu(\underline{R}(x, x_0) < \beta) \leq \mu(\limsup \overline{A}_i) = 0$$

- problem is reduced to one dimension

$$A_i := \{x \in \mathbb{S}^1 : \tau_{r_i}^{\alpha}(x) < (2r_i)^{-\beta}\}$$

- some care is required to get the good estimates for the measure of the intervals ...

Mixing reparametrizations - Yoccoz and Fayad

Take a vector with irrational coordinates (α_1, α_2). Let q_n, q_n' be the denominators of convergents.
We define a set of vectors Y by the following conditions:

$$q_n' \geq e^{3q_n}, \qquad q_{n+1} \geq e^{3q_n'}$$

Y is uncountable, dense set of zero Hausdorff dimension. Each coordinate is a Liouville number.

Theorem *(Fayad 02)*

For any torus translation by a vector in Y there exists a positive analytic function ϕ on $\mathbb{T}^3$ such that the reparametrization with speed $1/\phi$ of the suspension flow of the torus translation is mixing (with respect to Lebesgue measure).

Polynomial decay - Thm C

Theorem C *(Galatolo, P. 10)*

If a system on a manifold of dimension d has absolutely continous invariant measure with continuous and strictly positive density and polynomial decay of correlations (on Lipschitz observables) with exponent p, then for μ-almost every x

$$d \leq \limsup_{r \to 0} \frac{\log \tau_r(x, x_0)}{-\log r} \leq d + \frac{2d+2}{p}$$

This theorem and the invariance of hitting time under positive time reparametrization give a bound on decay of correlations for torus translations, depending on its arithmetical properties.
If one of the coordinate of the translation vector has type γ then the polynomial decay has exponent at most $\frac{2d+2}{\gamma - d}$.

- Thus Fayad's example has subpolynomial decay of correlations.