Game theory and statistical mechanics

Luc Rey-Bellet

University of Massachusetts Amherst

With **Sung-Ha Hwang** (University of Massachusetts Amherst)
and **Markos Katsoulakis** (University of Crete)

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• **Goal:** Explore (some of) the deep connections between game theory and statistical mechanics.

→ This is a two-way road, in particular game theory provides new model in nonequilbrium statistical mechanics.

• Traditional game theory is static and centered around the concepts of static equilibria between players (mostly in economics).

• "Recently" evolutionary game theory has emerged influenced by ideas from evolutionary biology which influenced back economics (and computer science, etc..) (cf. Hofbauer, Sigmund, P. Young, ...)

→ Population games: large population of agents

→ Dynamical selection of equilbria

→ Deterministic and stochastic dynamical model of the learning process ((bounded) rationality, error, imitation, etc...)

Normal form game

• Caveat: Only noncooperative game theory: no communication or coalition allowed. Also no "memory", anonymous players.

• 2 "players" called α and β

• Strategy sets: (finite) set $S\alpha$ and S_β.

• Payoff matrices: $A_\alpha(i,j)$ and $A_\beta(i,j)$, with $i \in S_\alpha$ and $j \in S_\beta$

$\rightarrow$ $A_\alpha(i,j) =$ payoff for player α if α chooses strategy i and β chooses strategy j.

• Mixed strategies p_α and p_β (probability vectors)

$\rightarrow p_\alpha = \{p_\alpha(i)\}_{i \in S_\alpha} \in \Delta_\alpha \Leftrightarrow p_\alpha(i) \geq 0$ and $\sum_i p_\alpha(i) = 1$.

Remark 0: Normal form is not a restrictive assumption.

Remark 1: "Strategy" and "payoffs" have many interpretations:

→ genotype in biology, payoff=fitness

→ "spin" in physics,

....

Remark 2: Population games

mixed strategy = randomized strategy, i.e., play i with prob. $p_\alpha(i)$

mixed strategy = population state, i.e., $p_\alpha(i)$ is the proportion of of players with strategy i.

Remark 3: Generalizations: We are discussing here only special cases (2 players, multilinear payoffs).... Can be done much more generally (p players, non linear games...)

Nash equilibrium

$\langle p_\alpha, A_\alpha p_\beta \rangle$ is the (expected) payoff for α if the players have mixed strategies p_α and p_β.

A pair of mixed strategies p_α^*, p_β^* is a Nash equilibrium if

$$\langle p_\alpha^*, A_\alpha p_\beta^* \rangle \geq \langle p_\alpha, A_\alpha p_\beta^* \rangle \quad \text{for all } p_\alpha$$

and

$$\langle p_\alpha^*, A_\beta p_\beta^* \rangle \geq \langle p_\alpha^*, A_\beta p_\beta \rangle \quad \text{for all } p_\beta$$

No player has any incentive to deviate if the other player do not change his strategy

Existence of NE follows from Brouwer's fixed point theorem.

Symmetric games

A normal form game is symmetric if player α and β have the same strategy sets, $S_\alpha = S_\beta$ and furthermore

$$A_\alpha(i,j) = A_\beta(j,i)$$

that is, the payoff for α if play α plays i against β playing j is the same as the payoff for player β to play if β plays i against α playing j.

Symmetric games describe a population of identical individuals.

p_α^* is a Nash equilibrium for a symmetric games if

$$\langle p_\alpha^*, A_\alpha p_\alpha^* \rangle \geq \langle p_\alpha, A_\alpha p_\alpha^* \rangle \quad \text{for all } p_\alpha$$

Example of Dynamics

Many different dynamics to model various behavior.

One of the the first and simplest dynamics for evolutionary games is the replicator dynamics.

For symmetric games let $p_t(i)$ = proportion of players using strategy i at time t.

$$\text{Replicator} \qquad \frac{d}{dt}p_t(i) = p_t(i)\left(Ap_t(i) - \langle p_t, Ap_t\rangle\right)$$

$\longrightarrow$ Biological interpretation: payoff $\equiv$ fitness, that is, the measure of reproductive succes. The rate of increase of strategy i is proportional to the excess fitness compared to the average fitness in the population.

$\longrightarrow$ Mean-field dynamics!

$\longrightarrow$ Interesting dynamical systems, sometimes chaotic (cf. Sparrow, van Strien for best response dynamics).

Spatial games

Not all economists think that space matters!

Spatially distributed players. For simplicity assume one type of player and thus use and use a symmetric game.

- Configuration space: Lattice $\Lambda \subset \mathbb{Z}^d$. At each site x one player with strategy $\sigma(x) \in S$. Configuration $\sigma_\Lambda = \{\sigma(x)\}_{x\in\Lambda} \in S^\Lambda$.

- Interaction: Choose weights $J(x-y)$ with

$$J(x-y) \geq 0 \quad \text{and} \quad \sum_y J(x-y) \approx 1$$

$J(x-y)$ is the weight that the player at x give to the player at y.

- Payoff for the player at x to pick strategy i

$$u(x,\sigma,i) = \sum_y J(x-y)A(i,\sigma(y))$$

Stochastic updating rules

Continuous time Markov process with generator

$$Lf(\sigma) = \sum_{x \in \Lambda} \sum_{i \in S} c(x,\sigma,i) \left[f(\sigma^{x,i}) - f(\sigma) \right]$$

Two typical examples of updating rates

- Logit – (generalized) Glauber dynamics

$$c_\beta(x,\sigma,i) = \frac{e^{\beta u(x,\sigma,i)}}{\sum_{k \in S} e^{\beta u(x,\sigma,k)}}$$

$\rightarrow$ The limit $\beta \rightarrow \infty$ gives the best response dynamics
(rationality=zero temperature)

• Imitation dynamics

$$c(x,\sigma_i) = w(x,\sigma,i)\, \mathsf{max}\, [u(x,\sigma,i) - u(x,\sigma,\sigma(x) \ , \ 0]$$

where $w(x,\sigma,i) = \sum_y J(x-y)\delta(i,\sigma(y))$

$\longrightarrow$ Pick a neighbour at random (according to the weights J)
$\longrightarrow$ Imitate him if his strategy is better than yours.
$\longrightarrow$ The rate of imitation is proportional to the excess payoff.

• Many other interesting choices of dynamics can be accomodated...

Mesocopic dynamics

- Kac's potential: assume

$$J_\gamma(x-y) = \gamma^d J(\gamma(x-y))$$

with $J(x) > 0$ and $\int J(x)dx = 1$ and let $\gamma \to 0$

- Rescale space to obtain a continuum limit.

$\mathbb{T}^d$ be the d-dimensional torus (mesoscopic domain)

$\mathbb{T}^{d,\gamma} := \gamma^{-1}\mathbb{T}^d \cap \mathbb{Z}^d$ (microscopic domain).

Other boundary conditions can be accomodated.

- Emprirical measure

$$\sigma \mapsto \pi^\gamma(\sigma; du, di) := \frac{1}{|\mathbb{T}^{d,\gamma}|} \sum_{x \in \mathbb{T}^{d,\gamma}} \delta_{(\gamma x, \sigma(x))}(dudi)$$

- Nice initial condition μ^γ, e.g., product measures with slowly varying parameters: $\mu^\gamma := \bigotimes_{x \in \mathbb{T}^{d,\gamma}} \rho_x$ and there exists a profile $f(u,i)$ such that

$$\rho_x(\{i\}) = f(\gamma x, i)$$

Mesoscopic equations

The mesoscopic strategy profiles $f(x,i) \in \mathcal{M}(\mathbf{T}^d \times S)$ where

$$\mathcal{M}(\mathbf{T}^d \times S) := \left\{ 0 \le f(x,i) \le 1, \ \sum\nolimits_i f(x,i) = 1 \text{ for all } x \in \mathbf{T}^d \right\}.$$

Theorem: Under suitable regularity conditions on the rates $c(x,\sigma,i)$ for every $t > 0$

$$\lim_{\gamma \to 0} \pi_t^\gamma(dx, di) = f_t(x,i)\, dx di \quad \text{in probability}$$

and f_t satisfies the following differential equation: for $u \in \mathbf{T}^d, i \in S$

$$\begin{aligned} \frac{\partial}{\partial t} f_t(x,i) &= \sum_{k \in S} \mathbf{c}(x,k,i,f) f_t(x,k) - f_t(x,i) \sum_{k \in S} \mathbf{c}(x,i,k,f) \\ f_0(x,i) &= f(x,i) \end{aligned}$$

Examples:

(a) Logit/generalized Glauber dynamics:

$$\frac{\partial}{\partial t} f_t(x,i) = \frac{\exp\left(\sum_{l\in S} a(i,l)\mathcal{J} * f_t(x,l)\right)}{\sum_{k\in S}\exp\left(\sum_{l\in S} a(k,l)\mathcal{J} * f_t(x,l)\right)} - f_t(x,i)$$

which generalizes the logit ODE of game theory and the equation in statistical mechanics (cf. Presutti & al.)

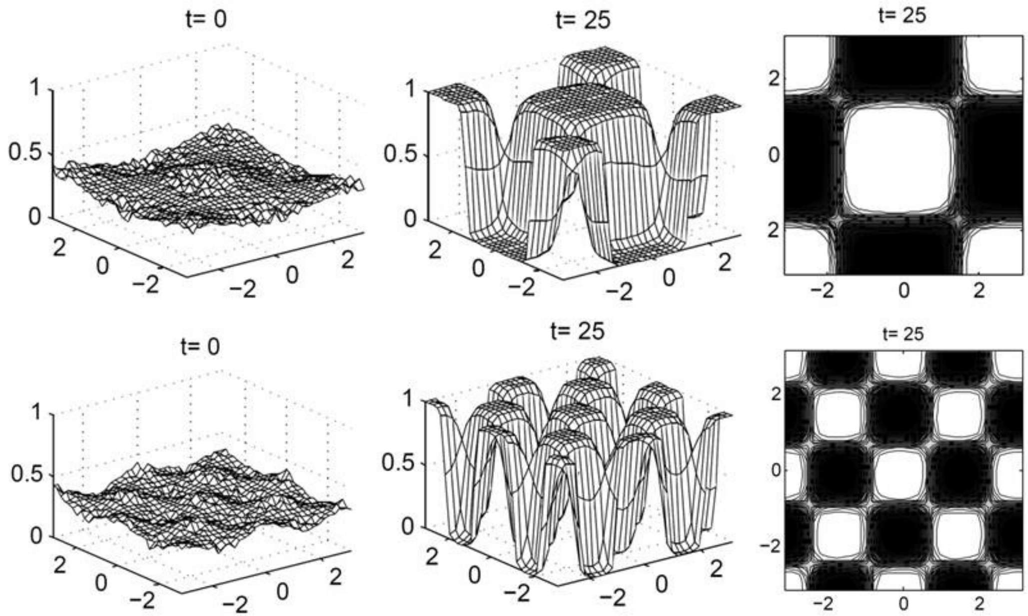
(b) Spatial replicator equation:

$$\begin{aligned} \frac{\partial}{\partial t} f_t(x,i) \quad &= \quad \sum_{k\in S}\Bigg[f(u,k)\mathcal{J} * f(x,i) F\left(\sum_{l\in S}(a(i,l)-a(k,l))\mathcal{J} * f_t(x,l)\right) \\ (1) \qquad & \qquad -f(x,i)\mathcal{J} * f(x,k) F\left(\sum_{l\in S}(a(k,l)-a(i,l))\mathcal{J} * f_t(x,l)\right)\Bigg] \end{aligned}$$

with $F(x) = \max\{x, 0\}$.

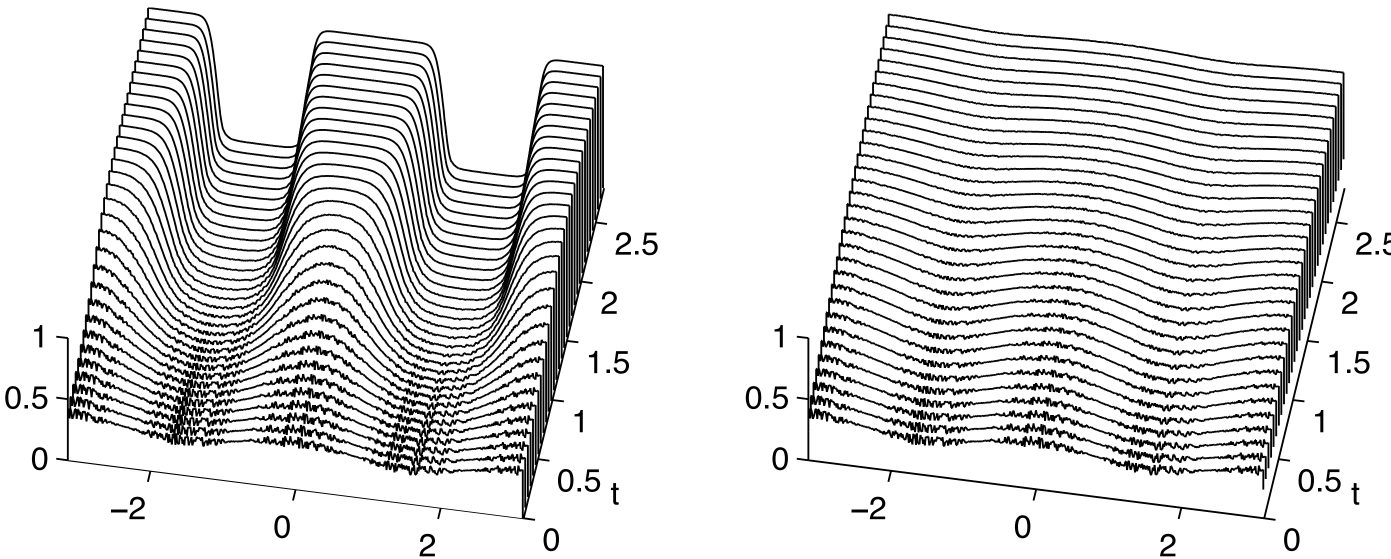
• Simple example $A = \begin{pmatrix} 20/3 & 0 \\ 0 & 10/3 \end{pmatrix}$. So-called coordination game, 3 Nash equilibrium $(1,0),(0,1),(1/3,2/3)$

• The Glauber dynamics is the Glauber dynamics for an Ising model with magnetic field.



$$d = 2,\ A = \begin{pmatrix} 20/3 & 0 \\ 0 & 10/3 \end{pmatrix}.\ p_0 = 1/3 + 1/10 \sin(x)\sin(y)$$

Imitative vs Logit/Glauber



Dimension 1, $A = \begin{pmatrix} 20/3 & 0 \\ 0 & 10/3 \end{pmatrix}$, $p_0 = 1/3 + 1/10 \sin(x)$

Many more problems (wip):

• Existence of travelling waves for imitative dynamics.

• Imitative dynamics has a non-compact attractor (?!). Segregation and convergence to the locallly preferred equilibrium and then freeze.....

• Study the statistics of fluctutations around the determinstic dynamics, (CLT+ LDP).

Decomposition of game & nonequilibrium stat mech

Two important class of games

- zero-sum games, i.e.,

$$A_\alpha(i,j) = -A_\beta(i,j)$$

My win is your lost

$\longrightarrow$ Nash equilibria (p^*_α, p^*_β) are found using *von Neumann minimax theorem*.

$$\langle p^*_\alpha, A_\alpha p^*_\beta \rangle = \max_{p_\alpha} \min_{p_\beta} \langle p_\alpha, A_\alpha p_\beta \rangle = \min_{p_\beta} \max_{p_\alpha} \langle p_\alpha, A_\alpha p_\beta \rangle$$

• potential games, i.e.,

$$A_\alpha(i,j) = A_\beta(i,j)$$

Partnership games: we share the spoil

$\longrightarrow$ Nash equilibria are determined by the *critical points* of the *potential function*

$$V(p_\alpha, p_\beta) = \langle p_\alpha, A_\alpha p_\beta \rangle = \langle p_\alpha, A_\beta p_\beta \rangle$$

on the simplex $\Delta_\alpha \times \Delta_\beta$ (Karush-Kuhn-Tucker conditions).

Invariance of Nash equilibria

• Passive game: If the payoff for α is $\begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_1 & \cdots & a_n \end{pmatrix}$ then all strategies gives equal payoff to α.

Fact: The Nash equilibria are invariant under the addition of passive game.

$$A_\alpha \mapsto A_\alpha + \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_1 & \cdots & a_n \end{pmatrix},$$

$$A_\beta \mapsto A_\beta + \begin{pmatrix} b_1 & \cdots & b_1 \\ \vdots & & \vdots \\ b_m & \cdots & a_m \end{pmatrix}$$

$\rightarrow$ This defines an equivalence relation $(A_\alpha, A_\beta) \sim (A'_\alpha, A'_\beta)$

Decomposition of games

$\mathcal{L} = \{(A_\alpha, A_\beta)\}$ with scalar product

$$Tr\left[\begin{pmatrix} 0 & A_\alpha \\ A_\beta^T & 0 \end{pmatrix}\begin{pmatrix} 0 & B_\alpha \\ B_\beta^T & 0 \end{pmatrix}^T\right]$$

be the (linear) space of games $\mathcal{L}$.

Define the potential game as

$$\mathcal{P} = \{(A_\alpha, A_\beta) \sim (A, A)\}$$

Orthogonal decomposition $\mathcal{L} = \mathcal{P} + \mathcal{N}$

$\mathcal{N}$ is the subspace of games generated by matching pennies games

$$A_\alpha = \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} A_\beta = \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix}$$

Decomposition of symmetric games

For $\mathcal{L}_{sym}$ the space of symmetric games (i.e., $A_\alpha = A_\beta^T$) we can decompose

$$\mathcal{L}_{sym} = \mathcal{P}_{sym} \oplus \mathcal{N}_{sym}$$

where

$\mathcal{P}_{sym}$ is the subspace of generalized symmetric potential games.

$\mathcal{N}_{sym}$ is subspace generated by generalized rock paper scissor

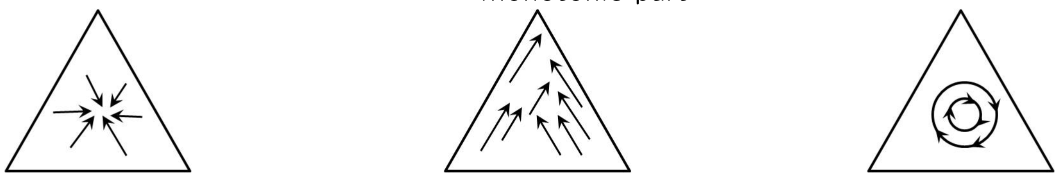
$$A_\alpha = \begin{pmatrix} & \vdots & & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & -1 & \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & & \vdots & \end{pmatrix}$$

Decomposition and replicator equation

Replicator equation $\dot{p}(i) = f_i(p) = p(i)\,(Ap(i) - \langle p, Ap\rangle)$

We have

$$f_i(p) = \underbrace{p_i((Sp)_i - x^T Sx)}_{\text{potential part}} + \underbrace{x_i(\eta_i - \sum_{l\neq 1}\eta_l x_l)}_{\text{monotonic part}} + \underbrace{x_i(Nx)_i}_{\text{conservative part}}$$



• Very useful to construct Liapunov functions and analyze the dynamics

Decomposition and Logit/generalized Glauber dynamics

$$c_\beta(x,\sigma,i) = \frac{e^{\beta u(x,\sigma,i)}}{\sum_{k\in S} e^{\beta u(x,\sigma,k)}}$$

Fact: The Markov process is reversible iff and only if the game is a generalized potential game with payoff $A \in \mathcal{P}_{sym}$. If $A \sim B$ with $B = B^T$ then we have

$\longrightarrow$ Invariant distribution $Z^{-1}e^{\beta H(\sigma)}$ with

$$H_\Lambda(\sigma) = \frac{1}{2}\sum_{x,y} J(x-y)B(\sigma(x),\sigma(y))$$

$\longrightarrow$ Ising Model

Flows and entropy production

• To the part of $C \in \mathcal{N}$ we can associate flows and entropy production, following Maes, Lebowitz-Spohn, etc...

• Decompose the path space of the process into a time-reversible and time irreversible part. Let $\{\sigma_s\}$ $0 \le s \le t$ be a path of the process with transitions at time t_l, at site x_l from strategy i_l to strategy j_l.

Let us define the functional

$$W(\{\sigma_s\}) = \sum_{l=1}^{n} \log \frac{c(x_l, \sigma(t_l), i_l)}{c(x_l, \sigma(t_l), j_l)}$$

We have

$$W(\{\sigma_s\}) = H(\sigma(t)) - H(\sigma(0)) + \sum_{l} J(i_l, j_l; \sigma(t_l))$$

where $J(i, j, \sigma)$ is determined entirely by the $C \in \mathcal{N}$.

We have almost surely

$$\frac{1}{t} W(\{\sigma_s\}_{0 \leq s \leq t}) \to EP(\mu) > 0$$

positivity of entropy production

steady local flows of strategies

One can also show

- Linear response theory and Kubo formula
- Fluctuation symmetry for the currents (Gallavotti-Cohen).