(Super)diffusive asymtotics for perturbed Lorentz or Lorentz-like processes

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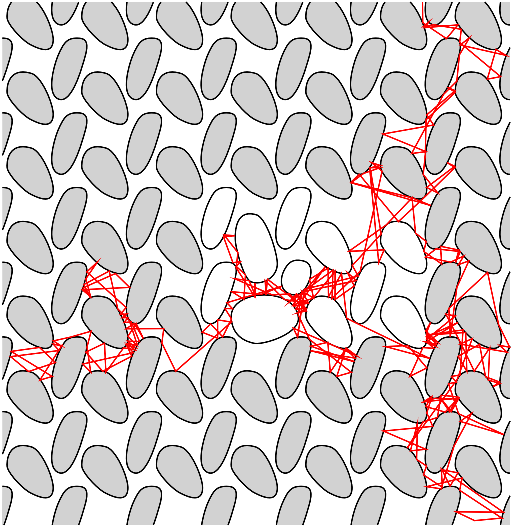
A Lorentz orbit



Notions and notations: Lorentz Process

Lorentz process - billiard dynamics (uniform motion + specular reflection) (Ω, T, μ)

- $\hat{Q} = \mathbb{R}^d \setminus \cup_{i=1}^{\infty} O_i$ is the configuration space of the Lorentz flow (the billiard table), where the closed sets O_i are pairwise disjoint, strictly convex with $\mathcal{C}^3-$smooth boundaries
- $\Omega = Q \times S_+$ is its phase space for the billiard ball map (where $Q = \partial\hat{Q}$ and S_+ is the hemisphere of outgoing unit velocities)
- $T : \Omega \to \Omega$ its discrete time billiard map (the so-called Poincaré section map)
- μ the T-invariant (infinite) Liouville-measure on Ω

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Notions and notations: Periodic Lorentz $\to$ Sinai Billiard

If the scatterer configuration $\{O_i\}_i$ is $\mathbb{Z}^d$-**periodic**, then the corresponding dynamical system will be denoted by $(\Omega_{per} = Q_{per} \times S_+, T_{per}, \mu_{per})$. It makes sense then to **factorize** it by $\mathbb{Z}^d$ to obtain a **Sinai billiard** $(\Omega_0 = Q_0 \times S_+, T_0, \mu_0)$. The natural projection $\Omega \to Q$ (and analogously for Ω_{per} and for Ω_0) will be denoted by π_q.

Finite horizon (FH) versus *infinite horizon* (∞H)

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Finite horizon (FH) versus *infinite horizon* (∞H)

Why are local perturbations/∞H interesting?

Local perturbations

- Lorentz, 1905: described the transport of conduction electrons in metals (still in the pre-quantum era). Natural to consider models with local impurities;
- Non-periodic models (M. Lenci, '96-, Sz., '08: Penrose-Lorentz process).

∞H

- Hard ball systems in the nonconfined regime have ∞H
- Crystals
- Non-trivial asymptotic behavior and new kinetic equ. (Bourgain, Caglioti, Golse, Wennberg, ...; '98-, Marklof-Strömbergsson, '08-).

Stochastic properties: Correlation decay

Let $f, g \quad M(= \Omega_0, \textit{billiard phase space}) \to \mathbb{R}^d$ be piecewise Hölder.

Definition

- With a given $a_n : n \geq 1$ (M, T, μ) has $\{a_n\}_n$-correlation decay if $\exists C = C(f,g)$ such that $\forall f, g$ Hölder and $\forall n \geq 1$

$$\left| \int_M f(g \circ T^n) d\mu - \int_M f d\mu \int_M g d\mu \right| \leq C a_n$$

- The correlation decay is exponential (EDC) if $\exists C_2 > 0$ such that $\forall n \geq 1$

$$a_n \leq \exp(-C_2 n).$$

- The correlation decay is stretched exponential (SEDC) if $\exists \alpha \in (0,1), C_2 > 0$ such that $\forall n \geq 1$

$$a_n \leq C_1 \exp(-C_2 n^{\alpha}).$$

Diffusively scaled variant

Definition

Assume $\{q_n \in \mathbb{R}^d | n \geq 0\}$ is a random trajectory. Then its *diffusively scaled variant* $\in C[0,1]$ $\;$ (*or* $\;\in C[0,\infty]$) is defined as follows: for $N \in \mathbb{Z}_+$ denote
$W_N(\frac{j}{N}) = \frac{q_j}{\sqrt{N}} \qquad (0 \leq j \leq N \;\; or \;\; j \in \mathbb{Z}_+)$ and define otherwise $W_N(t)(t \in [0,1] \;\; or \;\; \mathbb{R}_+)$ as its piecewise linear, continuous extension.

E. g. $\kappa(x) = \pi_q(Tx) - \pi_q(x) : M \to \mathbb{R}^d$, the free flight vector of a Lorentz process.

From now on $q_n = q_n(x) = \sum_{k=0}^{n-1} \kappa(T^k x), \quad n = 0, 1, 2, \ldots$ is the Lorentz trajectory.

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Stochastic properties: CLT & LCLT

Definition

- *CLT and Weak Invariance Principle*

$$W_N(t) \Rightarrow W_{\mathcal{D}^2}(t),$$

  the Wiener process with a non-degenerate covariance matrix $\mathcal{D}^2 = \mu_0(\kappa_0 \otimes \kappa_0) + 2\sum_{j=1}^{\infty} \mu_0(\kappa_0 \otimes \kappa_n)$.

- Local CLT Let x be distributed on Ω_0 according to μ_0. Let the distribution of $[q_n(x)]$ be denoted by Υ_n. There is a constant $\mathbf{c}$ such that

$$\lim_{n\to\infty} n\Upsilon_n \to \mathbf{c}^{-1} I$$

  where I is the counting measure on the integer lattice $\mathbb{Z}^2$ and $\to$ stands for vague convergence.
  In fact, $\mathbf{c}^{-1} = \frac{1}{2\pi\sqrt{\det \mathcal{D}^2}}$.

2D, Periodic case: Some Results

		SEDC	EDC	CLT	LCLT
B-S, '81	M-partitions	X		X	
B-Ch-S, '91	M-sieves	X		X	
Y, '98	M-towers		X	X	
Sz-V, '04					X

SEDC - Stretched Exponential Decay of Correlations
EDC - Exponential Decay of Correlations
CLT - Central Limit Theorem
LCLT - Local CLT

Locally perturbed FH Lorentz

- Sinai's problem, '81: locally perturbed FH Lorentz
- Sz-Telcs, '82: locally perturbed SSRW for $d = 2$ has the same diffusive limit as the unperturbed one

  Idea: local time is $O(\log n)$ thus the $\sqrt{n}$ scaling eates perturbation up

  Method:
  - there are $\log n$ time intervals spent at perturbation
  - couple the intervals spent outside perturbations to SSRW

Locally perturbed FH Lorentz

Dolgopyat-Sz-Varjú, 09: locally perturbed FH Lorentz has the same diffusive limit as the unperturbed one
Method: Martingale method of Stroock-Varadhan

Tools:

- Chernov-Dolgopyat, 05-09:
  - standard pairs
  - growth lemma
  - Young-coupling
- Sz-Varjú, 04: local CLT for periodic FH Lorentz
- Dolgopyat-Sz-Varjú, 08: recurrence properties of FH Lorentz (extensions of Thm's of Erdős-Taylor and Darling-Kac from SSRW to FH Lorentz)

∞H periodic Lorentz

Reminder: $\kappa(x) = \pi_q(Tx) - \pi_q(x) : M \to \mathbb{R}^2$, the free flight vector of a Lorentz process.

$q_n = q_n(x) = \sum_{k=0}^{n-1} \kappa(T^k x)$ is the Lorentz trajectory.

Now: for $N \in \mathbb{Z}_+$ denote

$$W_N\left(\frac{j}{N}\right) = \frac{q_j}{\sqrt{N \log N}} \qquad (0 \leq j \leq N \quad or \quad j \in \mathbb{Z}_+)$$

and define otherwise $W_N(t)(t \in [0,1] \quad or \quad \mathbb{R}_+)$ as its piecewise linear, continuous extension.

∞H periodic Lorentz

- Bleher, '92:
  - $\mathbb{E}|\kappa(x)|^2 = \infty$
  - $\mathbb{E}|\kappa(x)\kappa(T^n x)| < \infty$ *if* $|n| \geq 1$.
  - Heuristic arguments for $\sqrt{N \log N}$ scaling.
- Sz-Varjú, 07:
  - Rigorous proof for Bleher's conjecture (method: Young's towers & Fourier transform of P-F operator)
  - Moreover local limit law & Recurrence
  - Exact form of the limiting covariance
- Melbourne, '08, $O(1/t)$ corr. decay rate for the flow
- Chernov-Dolgopyat, '10: EDC & global LT for κ (method: Ch-D's standard pairs & Bernstein's method of freezing)

Martingale approach
à la Stroock-Varadhan

Since the limiting process is a Brownian motion, it is characterized by the fact that

$$\phi(W(t)) - \frac{1}{2}\int_0^t \sum_{ab=1,2} \sigma_{ab} D_{ab}\phi(W(s))ds \qquad (1)$$

is a martingale for C^2-functions of compact support.

By Stroock-Varadhan it suffices to show that — the limiting process $\tilde{W}(t)$ of any convergent subsequence of the processes in question — the process

$$\phi(\tilde{W}(t)) - \frac{1}{2}\int_0^t \sum_{ab=1,2} \sigma_{ab} D_{ab}\phi(\tilde{W}(s))ds \qquad (2)$$

is a martingale for C^2-functions of compact support.

Locally perturbed FH 1

Let ϕ be a smooth function of compact support. Denote $n = Nt$ and choose a small $\alpha > 0$. Let $L = N^{\alpha}$. Let $m_p = pL + z$ $(p \in \mathbb{Z}_+)$ where z will be chosen later. Denote

$$\Delta_j = q_{j+1} - q_j.$$

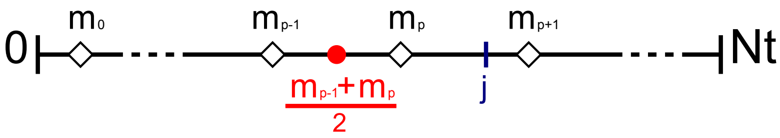
By summing up second order Taylor-expansions of $\phi\left(\frac{q_{j+1}}{\sqrt{N}}\right) - \phi\left(\frac{q_j}{\sqrt{N}}\right)$:

$$\phi\left(\frac{q_{m_{p+1}}}{\sqrt{N}}\right) - \phi\left(\frac{q_{m_p}}{\sqrt{N}}\right) =$$

$$\sum_{j=m_p+1}^{m_{p+1}} \frac{1}{\sqrt{N}} \left\langle D\phi\left(\frac{q_j}{\sqrt{N}}\right), \Delta_j \right\rangle + \frac{1}{2} \sum_{j=m_p+1}^{m_{p+1}} \frac{1}{N} \left\langle D^2\phi\left(\frac{q_j}{\sqrt{N}}\right)\Delta_j, \Delta_j \right\rangle$$

$$+O(LN^{-3/2}).$$

Raster



$$n = Nt \qquad L = N^{\alpha} \quad (\alpha > 0) \qquad m_p = pL + z \qquad 0 \le z = m_0 < L$$

Locally perturbed FH 2

Next

$$\phi\left(\frac{q_{m_{p+1}}}{\sqrt{N}}\right)-\phi\left(\frac{q_{m_p}}{\sqrt{N}}\right)=\sum_{j=m_p+1}^{m_{p+1}}\frac{1}{\sqrt{N}}\left\langle D\phi\left(\frac{q_{m_{p-1}}}{\sqrt{N}}\right),\Delta_j\right\rangle$$

$$+\frac{1}{N}\left[\frac{1}{2}\sum_{j=m_p+1}^{m_{p+1}}\left\langle D^2\phi\left(\frac{q_{m_{p-1}}}{\sqrt{N}}\right)\Delta_j,\Delta_j\right\rangle\right.$$

$$\left.+\sum_{m_{p-1}<k<j}\left\langle D^2\phi\left(\frac{q_{m_{p-1}}}{\sqrt{N}}\right)\Delta_k,\Delta_j\right\rangle\right]$$

$$+O(L^2N^{-3/2}).$$

Standard pair

- A connected smooth curve $\gamma \subset \Omega_0$ is called an *unstable curve* if at every point $x \in \gamma$ the tangent space $\mathcal{T}_x\gamma$ belongs to the unstable cone $\mathcal{C}_x^u$.
- A *standard pair* is a pair $\ell = (\gamma, \rho)$ where γ is a homogeneous curve and ρ is a homogeneous density on γ.

Growth lemma, Ch-D

Theorem

- *If* $\ell = (\gamma, \rho)$ *is a standard pair, then*

$$\mathbb{E}_\ell(A \circ T_0^n) = \sum_\alpha c_{\alpha n} \mathbb{E}_{\ell_{\alpha n}}(A)$$

where $c_{\alpha n} > 0$, $\sum_\alpha c_{\alpha n} = 1$ *and* $\ell_{\alpha n} = (\gamma_{\alpha n}, \rho_{\alpha n})$ *are standard pairs where* $\gamma_{\alpha n} = \gamma_n(x_\alpha)$ *for some* $x_\alpha \in \gamma$ *and* $\rho_{\alpha n}$ *is the pushforward of* ρ *up to a multiplicative factor.*

- *If* $n \geq \beta_3 |\log \mathrm{length}(\ell)|$, *then*

$$\sum_{\mathrm{length}(\ell_{\alpha n}) < \varepsilon} c_{\alpha n} \leq \beta_4 \varepsilon.$$

Moment asymptotics, Ch-D

Theorem

Let ℓ be a standard pair, A a Hölder function. Take n such that $|\log \mathrm{length}(\ell)| < n^{1/2-\delta}$.

- $\exists C_1, C_2 > 0 \;\; \theta < 1$ *s. t. if* $n > C_1 |\log \mathrm{length}(\ell)|$, *then*

$$\left| \mathbb{E}_\ell(A \circ T_0^n) - \int A d\mu_0 \right| \leq C_2 \theta^n$$

- *Let* $A, B \in \mathcal{H}$ *with zero mean. Denote* $A_n(x) = \sum_{j=0}^{n-1} A(T_0^j x)$. *Then*

$$\mathbb{E}_\ell(A_n B_n) = n\sigma_{A,B} + O(|\log^2 \mathrm{length}(\ell)|)$$

where

$$\sigma_{A,B} = \sum_{j=-\infty}^{\infty} \int A(x) B(T_0^j x) d\mu_0(x).$$

A Markov-Taylor expansion

$$\phi\left(\frac{q_{m_{p+1}}}{\sqrt{N}}\right)-\phi\left(\frac{q_{m_p}}{\sqrt{N}}\right)=\sum_{j=m_p+1}^{m_{p+1}}\frac{1}{\sqrt{N}}\left\langle D\phi\left(\frac{q_{m_{p-1}}}{\sqrt{N}}\right),\Delta_j\right\rangle$$

$$+\frac{1}{N}\left[\frac{1}{2}\sum_{j=m_p+1}^{m_{p+1}}\left\langle D^2\phi\left(\frac{q_{m_{p-1}}}{\sqrt{N}}\right)\Delta_j,\Delta_j\right\rangle\right.$$

$$\left.+\sum_{m_{p-1}<k<j}\left\langle D^2\phi\left(\frac{q_{m_{p-1}}}{\sqrt{N}}\right)\Delta_k,\Delta_j\right\rangle\right]$$

$$+O(L^2N^{-3/2}).$$

Decompositions

Consider the Markov decomposition

$$\mathbb{E}_\ell(A \circ T^{m_p}) = \sum_\alpha c_\alpha \mathbb{E}_{\ell_\alpha}(A \circ T^{(m_{p-1}+m_p)/2}) = \mathcal{T}_1 + \mathcal{T}_2$$

where $A = \phi\left(\frac{q_L}{\sqrt{N}}\right) - \phi\left(\frac{q_0}{\sqrt{N}}\right)$, *and*
$\mathcal{T}_1$ is the sum over α such that $|q_{m_{p-1}}| \geq KL$ and
$\mathcal{T}_2$ is the sum over α such that $|q_{m_{p-1}}| < KL$.

To estimate $\mathcal{T}_1$ split it $\mathcal{T}_1' + \mathcal{T}_1''$ where $\mathcal{T}_1'$ (the main term!) contains αs with $\text{length}(\ell_\alpha) > N^{-100}$.

$\mathcal{T}_1''$ can be handled by using the growth lemma.

A priori bound and main term

$\mathcal{T}_2$ can be handled by using an a priori bound

Lemma

Fix S, a finite collection of scatterers. There is a constant $\tilde{K}$

$$\mathbb{E}_\ell(\mathrm{Card}(j \le n : q_j \in S)) \le \tilde{K} \log^{1+\xi} N$$

where $\xi > 0$.

For the main term use the Markov-Taylor expansion:

$$\mathcal{T}_1' = \mathcal{T}_{lin} + \mathcal{T}_{quad} + \mathcal{T}_{rem}$$

Its terms can be handled by using the Markov decomposition and the moment asymptotics.

Thm for locally perturbed FH, D-Sz-V, '09

Theorem

For finite modifications of the FHLP, as $N \to \infty$, $W_N(t) \Rightarrow W_{\Sigma^2}(t)$ (weak convergence in $C[0,\infty]$), where $W_{\Sigma^2}(t)$ is the Brownian Motion with the non-degenerate covariance matrix Σ^2. The limiting covariance matrix coincides with that for the unmodified periodic Lorentz process.

Geometry & Probability

- Corridors
- Jump length for discrete version: $\mathbb{P}(\Delta_j = k) \sim const. \frac{1}{k^3}$
- By using truncation à la Ch-D: $\hat{\Delta}_k = Min\{\Delta_k, \sqrt{N} \log^\beta N\}$

$$\mathbb{E}|\hat{\Delta}_k|^h = O(N^{\frac{h-2}{2}} \log^{\beta(h-2)} N) \qquad if \;\; h \geq 3$$

$= O(\log N)$ for $h = 2$ and $= O(1)$ for $h \leq 1$.

- Paulin-Sz., '09: for random walks
  - a with jumps belonging to the non-standard domain of attraction of Gaussian
  - and with local impurities

  the same limit behavior holds as for the periodic RW
- Nándori, '09: if impurity is in 0, but it also acts when flying through, then 'local time' for 0 is $O(n^{1/6})$.

Martingale method for periodic Lorentz

- growth lemma holds (in fact, also for perturbed Lorentz)
- moment estimates and EDC hold by Ch-D
- apply the Markov-Taylor expansion to $\hat{q}_j = \Sigma_{k=1}^{j} \hat{\Delta}_k$
- the error terms can be handled by using the bounds on $\mathbb{E}|\hat{\Delta}_k|^h$, and some Höldering;
- Result: third proof for global LT for ∞H periodic Lorentz.