A heat conduction model with localized billiard balls and weak interaction forces

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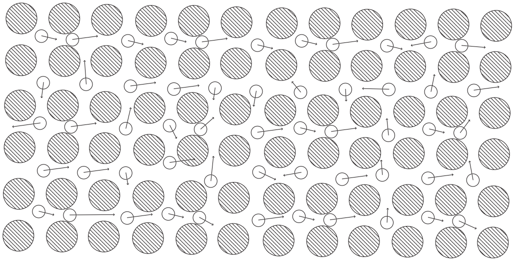
Preliminary remarks

- Motivation: Gaspard-Gilbert model
- work in progress – more phenomena than complete proofs
- Carlangelo Liverani will (also) talk about something very similar tomorrow
- I apologize for my first beamer presentation

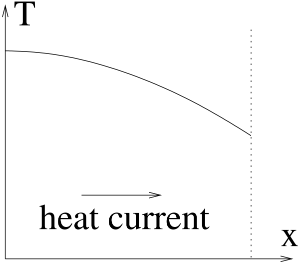
The model

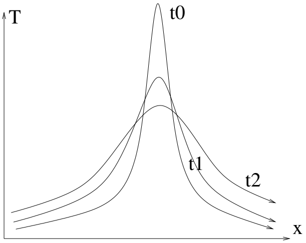


- particles cannot collide
- neigbours interact via some potential
- (compare Gaspard-Gilbert)

Goal

Understand heat conduction





- big system + thermostats, *or*
- infinite system + nenequilibrium initial conditions

Temperature defined e.g. as

- expectation of energy, *or*
- $T = 1/\beta$ if *energy* $\sim e^{-\beta E}$

Dreams

Dreams: Fourier's law

$$\partial_t T(t,x) = -\nabla_x J(t,x)$$

$$J(t,x) = D(T(t,x))\nabla_x T(t,x)$$

- not obviously true: models with "not enough nonlinearity" exhibit anomalous heat conduction
- out of reach for this system at the moment

What I think can be done

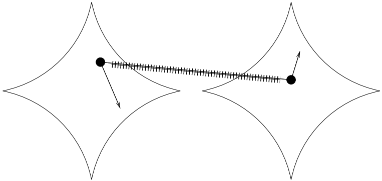
When there is hope: **weak coupling**: *force* $=\varepsilon F$

Programme:

- **Step 1**: $\begin{array}{c}\text{finite size}\\ \text{dynamical system}\end{array} \xrightarrow{\varepsilon\searrow 0,\ t\rightsquigarrow t/\varepsilon^2} \{E_i(t)\}_{i\in\Lambda\subset\mathbb{Z}^2}$

  Markov process (=interacting particle system)
  (hyperbolic dynamical systems problem)
- **Step 2**: hydrodynamics of the interacting particle system
  (problem for stochastics people)

Step 0: two particles



$$U(x_1, x_2) = \varepsilon V(|x_1 - x_2|) + \text{billiard reflections}$$

$$\begin{aligned}
\dot{E}_1^{mic,\varepsilon} &= v_1^\varepsilon \varepsilon F(x_1^\varepsilon - x_2^\varepsilon) = \varepsilon P_1(\text{fast variables}) \\
\dot{E}_2^{mic,\varepsilon} &= -v_2^\varepsilon \varepsilon F(x_1^\varepsilon - x_2^\varepsilon) = \varepsilon P_2(\text{fast variables}) \\
\dot{x}_1^\varepsilon &= v_1^\varepsilon \\
\dot{x}_2^\varepsilon &= v_2^\varepsilon \\
\left.\begin{aligned}
\dot{v}_1^\varepsilon &= 0 + \varepsilon F(x_1^\varepsilon - x_2^\varepsilon) \\
\dot{v}_2^\varepsilon &= 0 - \varepsilon F(x_1^\varepsilon - x_2^\varepsilon)
\end{aligned}\right\} &+ \text{billiard reflection boundary cond.}
\end{aligned}$$

(F=force acting on particle 1; P=power of force)

Why t/ε^2?

$$\dot{E}_1^{mic,\varepsilon} = v_1^\varepsilon \varepsilon F(x_1^\varepsilon - x_2^\varepsilon) = \varepsilon P_1(\text{fast variables})$$

Attempt 1: scale $t \rightsquigarrow t/\varepsilon$. That is, set

$$E_1^\varepsilon(t) := E_1^{mic,\varepsilon}(t/\varepsilon).$$

Not hard to guess:

$$E_1^\varepsilon(t) \overset{\varepsilon\searrow 0}{\Longrightarrow} E_1(t) \textbf{ deterministic},$$

such that

$$\dot{E}_1(t) = b(E_1(t)),$$

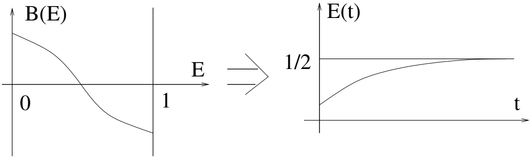
where

$$b(E) = \int_{E_1=E} P_1(fast)\,\mathrm{d}\mu_{inv}^{fast}.$$

Why not t/ε?

$$\dot{E}_1(t) = b(E_1(t))$$

One could think that



,

but no. CRUCIAL FACT: In our model $b(E) \equiv 0$:
on this time scale nothing happens.
(This is good news: we prefer a (limiting) model with a physically realistic invariant measure.) (compare Bricmont-Kupiainen)

Yes, t/ε^2.

Attempt 2: scale $t \rightsquigarrow t/\varepsilon^2$. That is, set

$$E_1^\varepsilon(t) := E_1^{mic,\varepsilon}(t/\varepsilon^2).$$

Now much better:

$$E_1^\varepsilon(t) \overset{\varepsilon \searrow 0}{\Longrightarrow} E_t \textbf{ nondeterministic Markov},$$

indeed

$$\mathbb{E}((E_{t+\mathrm{d}t} - E_t)^2 \mid E_{\leq t}) \approx \left[\int_{-\infty}^{\infty} \int_{E_1 = E_t} P_1(\Phi^\tau \mathit{fast}) P_1(\mathit{fast})\, \mathrm{d}\mu_{inv}^{fast}\, \mathrm{d}\tau\right] \mathrm{d}t$$

$=: \sigma^2(E_t)\,\mathrm{d}t \gneq 0$, where Φ^τ is the *uncoupled* flow of the two particles. (remember CLT and Green-Kubo)

The limiting process

Moments:

- $\mathbb{E}((E_{t+\mathrm{d}t} - E_t)^2 \mid E_{\leq t}) \approx \sigma^2(E_t)\,\mathrm{d}t =$ Green-Kubo
- $\mathbb{E}(E_{t+\mathrm{d}t} - E_t \mid E_{\leq t}) \approx b(E_t)\,\mathrm{d}t =$ much uglier formula.

In the language of stochastic processes:

$$Lf = \frac{1}{2}\sigma^2\nabla^2 f + b\nabla f$$

$$\mathrm{d}E_t = b(E_t)\,\mathrm{d}t + \sigma(E_t)\,\mathrm{d}W_t$$

About the method

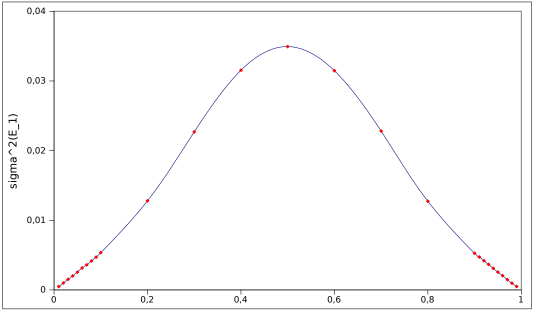
Keywords of proof (motivated by Chernov-Dolgopyat)

- *standard pair*: measure concentrated on a (short) unstable manifold $\approx$ conditioning on the entire past
- *coupling*: to show that any standard pair evolves (under the dynamics) quickly into something close to the invariant measure
- separation of time scales:
    - the fast system equilibrates while the energies are nearly constant
    - technically: evolution of unstable manifolds (standard pairs) under the true dynamics can be well approximated by the evolution under the "free" dynamics (compare G-G)
- *martingale method*: Show convergence of expectations, as $\varepsilon \searrow 0$, of expressions composed of test functions and the process $E_1^\varepsilon(t)$, and get convergence to the Markov process.

The problem of low energies

Q: What if a particle (nearly) stops?

A: This does not happen.



(+ we know $b = (\frac{1}{2}\sigma^2)'$, and think of the square Bessel process)

Step 1: finitely many particles

$$\mathrm{d}E_t^i = \sum_j \Big(b(E_t^i, E_t^j)\,\mathrm{d}t + \sigma(E_t^i, E_t^j)\,\mathrm{d}W_t^{ij} \Big)$$

- the sum runs over all neighbours j of i
- the W_t^{ij} are Wiener processes, independent for different edges, but $W_t^{ij} = -W_t^{ji}$.

In words:

- On every edge of the lattice there sit independent Wiener processes governing the energy transfer through the edge,
- the drift b and the diffusion coefficient σ of the transfer through the edge depends only on the energies at the sites connected.

Step 1: finitely many particles

$$\mathrm{d}E_t^i = \sum_j \Big(b(E_t^i, E_t^j)\,\mathrm{d}t + \sigma(E_t^i, E_t^j)\,\mathrm{d}W_t^{ij} \Big)$$

Symmetries:

- $b(x, y) = -b(y, x)$ and $\sigma(x, y) = \sigma(y, x)$: conservation of energy
- b can be expressed in terms of σ, which corresponds to the universality of the invariant measure, inherited from the invariant (Liouville) measure of the Hamiltonian system.
- σ is homogeneous in the total energy involved: $\sigma(Ex, Ey) = E^{1/4}\sigma(x, y)$. This is extremely useful in the study of the hydrodynamic limit.

Step 3: heat conduction

Step 3: Heat conduction in the interacting particle system

- Proving anything is probably difficult:
  - I know next to nothing about the topic (as of 02.06.2010)
  - the system is not gradient ($\Rightarrow$ no entropy method(?))
  - The system is not a small perturbation of something well understood ($\Rightarrow$ no renormalization method(?))
- Still it's possibly easier than Gaspard-Gilbert: energy fluxes are much smaller
- Heuristically the situation is clear: if we believe non-anomallous heat conduction, then from the scaling properties

$$D(T) = const\ T^{-3/2}$$

Conclusion

This is a great model:

$$D(T) = const\ T^{-3/2}$$

Compare

- Gaspard-Gilbert: $D(T) = const\ T^{+1/2}$
- experimental data (silicon, high temperature): $D(T) = const\ T^{-1.3}$

There is a lot to be done.

thanks

Thank you for your attention.