Contact Anosov flow and FBI transform

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Contact Anosov Flow

Let (M^{2d+1}, α) be a closed manifold with a contact 1 form α.

A C^∞ flow $F^t : (M, \alpha) \to (M, \alpha)$ is called a contact Anosov flow if it is an Anosov flow and preserves α.

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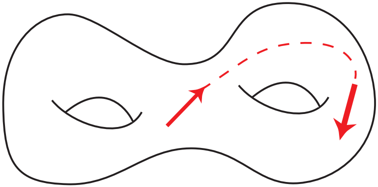

Transfer operators

We consider the one-parameter family of transfer operators

$$\mathcal{L}^t : C^\infty(M) \to C^\infty(M), \qquad \mathcal{L}u(x) = g^t(x)u(F^t(x))$$

where $g^t : M \to \mathbb{C}$ is C^∞ multiplicative cocycle.

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Spectral properties of the transfer operator $\mathcal{L}^t$ (when it acts on an appropriate function space).

We will assume $g^t \equiv 1$ for simplicity, in most places.

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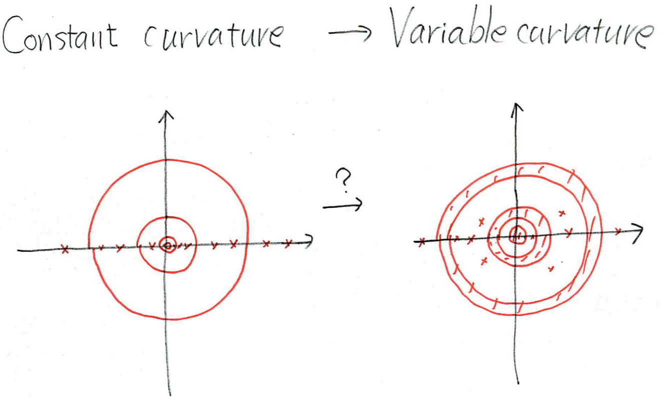

The action $\mathcal{L}^t$ viewed through Fourier transform

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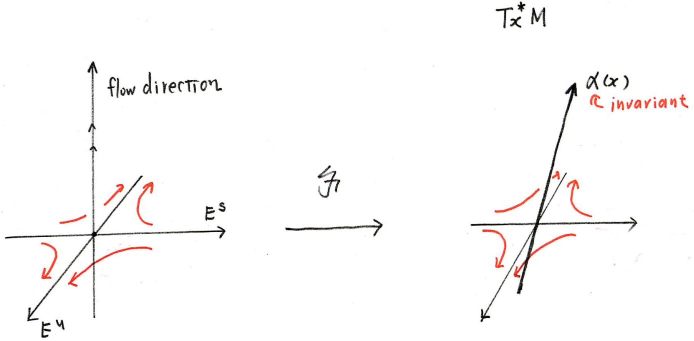

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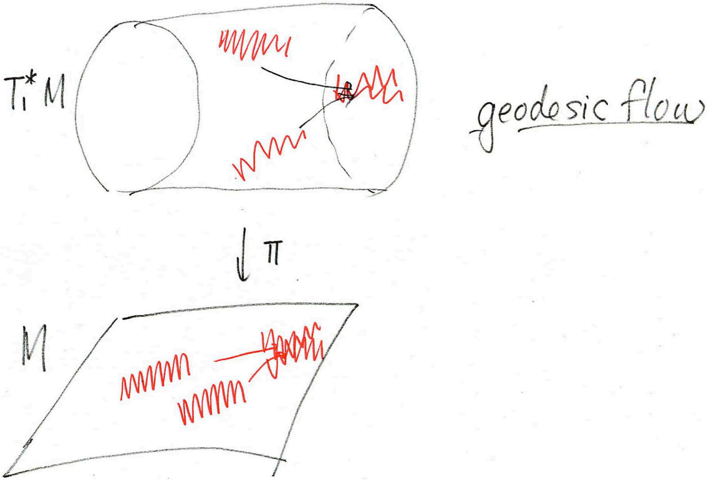
Is (techniques in) semi-classical analysis useful in the analysis of contact Anosov flows?



T_1^*M
geodesic flow
π
M

Main result

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Theorem

There exists a Hilbert space (Anisotropic Sobolev space) $\boldsymbol{H} \subset \mathscr{D}'(\boldsymbol{M})$ *such that* $\rho_{\mathrm{ess}}(\mathcal{L}^t) = \Lambda^t$ *where*

$$\Lambda = \lim_{t\to\infty} \left(\sup_{x\in M} \frac{1}{\sqrt{\det DF^t|_{E^u}}} \right)^{1/t}$$

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In what follows, we describe a proof of this theorem using (partial) FBI transform, a tool from semiclassical analysis.

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Conjecture (Subject of work in progress with F. Faure)

$$\text{Ess-spec}(\mathcal{L}^t|_H) \subset \{\underline{\Lambda}^t \le |z| \le \overline{\Lambda}^t\} \cup \{|z| \le \lambda^{-t}\cdot\overline{\Lambda}^t\}$$

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Remark

A particularly interesting case will be $g^t = \sqrt{\det DF^t|_{E^u}}$, where $\overline{\Lambda} = \underline{\Lambda} = 1$.

Sketch of the proof: A local model of $\mathcal{L}^t$

Consider the Euclidean space $\mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R}^2$ equipped with the standard contact form

$$\alpha_0(t, x_+, x_-) = dt + (x_+ dx_- - x_- dx_+)/2$$

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And let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map (preserving α_0)

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Below we consider the operator $Lu = u \circ A$ as a localized and simplified model of the transfer operator for contact Anosov flow.

FBI (Fourier-Bros-Iagolnitzer) transform

Let us consider a Gaussian wave packet

$$\phi_{x,\xi}(y) := \alpha_{n,k} \exp(i(y - x/2)\xi - |y - x|^2/2) \qquad (x, y, \xi \in \mathbb{R}^2)$$

where $\alpha_{n,k} = 2^{-1}\pi^{-3/2}$.

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where $\alpha_{n,k} = 2^{-1}\pi^{-3/2}$. We define $\mathcal{T} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^4)$ by

$$\mathcal{T}(x, \xi) = \int \overline{\phi_{x,\xi}(y)} u(y) dy$$

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Then we have $\mathcal{T}^*\mathcal{T} = \mathbf{Id}$ and hence

$$u(y) = \int \mathcal{T}u(x, \xi) \cdot \phi_{x,\xi}(y) dx d\xi \quad \text{(wavepacket decomposition)}$$

Partial FBI transform

Partial FBI transform $\mathbb{T} : L^2(\mathbb{R} \oplus \mathbb{R}^2) \to L^2(\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R})$ is

$$\begin{aligned}\mathbb{T}u(x, \xi, \nu) &= \int \overline{e^{i\nu t}\phi^{(\nu)}_{x,\xi}(y)}u(t, y)dtdy \\ &= \int \overline{\phi^{(\nu)}_{x,\xi}(y)} \left(\int e^{-i\nu t}u(t, y)dt \right) dy\end{aligned}$$

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where

$$\phi^{(\nu)}_{x,\xi}(y) := \alpha_{n,k} \cdot \nu \cdot \exp(i(y - x/2)\xi - \nu|y - x|^2/2)$$

"Fourier transform in $\mathbb{R}$" + "(scaled) FBI tranform in $\mathbb{R}^2$"

Anisotropic Sobolev space

Anisotropic Sobolev space $\boldsymbol{H}$ is defined as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_H = \|W(x,\xi,\nu)\cdot \mathbb{T}u(x,\xi,\nu)\|_{L^2(\mathbb{R}^2\oplus\mathbb{R}^2\oplus\mathbb{R})}$$

where the weight function W is defined so that, for $\gamma \gg 1$,

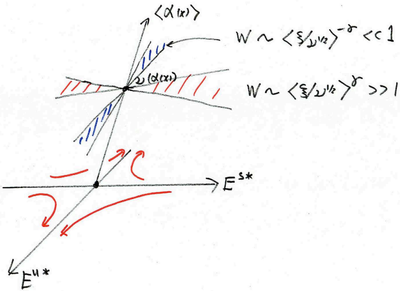
$$W(x,\nu\cdot\alpha_0(0,x)+(\xi,0)) \sim \begin{cases} \langle \nu^{-1/2}\xi\rangle^{\gamma}, & \text{if } \xi \text{ is in the stable cone;} \\ \langle \nu^{-1/2}\xi\rangle^{-\gamma}, & \text{if } \xi \text{ is in the unstable cone;} \end{cases}$$

Construction of the weight function W

More precisely, we set $w(\xi) = \langle \xi \rangle^{\mu(\xi/|\xi|)}$ where

$$\mu(\xi) = \begin{cases} \gamma, & \text{if } \xi \in \mathbb{R}^2 \text{ is in the stable cone;} \\ -\gamma, & \text{if } \xi \in \mathbb{R}^2 \text{ is in the unstable cone.} \end{cases}$$

and put $W(x, \xi, \nu) = w(\nu^{-1/2}((\xi, \nu) - \nu\alpha_0(x)))$.



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Remark

- Nonlinearity induces only small perturbations (and does not affect the essential spectral radius).
- If we apply such estimate for $\mathcal{L}^t$ on local charts and let $t \to \infty$, we get the main theorem.

The lift of $\boldsymbol{L}$ with respect to $\mathbb{T}$

To prove the theorem, we consider the lift of $\boldsymbol{L}$:

$$\widehat{\boldsymbol{L}} = \mathbb{T} \circ \boldsymbol{L} \circ \mathbb{T}^* : L^2(\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}) \to L^2(\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R})$$

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such that the following diagram commutes

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It is then enough to show

$$\left\| \widehat{L} : L^2(\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}, W) \to L^2(\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}, W) \right\| \leq C\lambda^{-1/2}$$

We may write $\widehat{L}$ as

$$\widehat{L} = \det((\mathrm{Id} + A^* A)/2)^{1/2} \cdot \mathbb{P} \circ \widetilde{L} \circ \mathbb{P}$$

where $\widetilde{L}$ is pull-back by $(A|_{\{0\}\times\mathbb{R}^2}) \oplus {}^t A^{-1}$ and

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$\mathbb{P}$ is the orthogonal projection to the image $\mathbb{T}(L^2(\mathbb{R}^2))$ and with kernel

$$K((x,\xi,\nu),(x',\xi',\nu')) = \delta(\nu - \nu') \cdot e^{i\Omega(x,\xi;x',\xi')/2 - |(x,\xi)-(x',\xi')|^2/4}$$

where $\Omega = dx \wedge d\xi$.

We introduce a (magical) change of variable (due to F. Faure)

$$\Phi : \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R} \to \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R},$$

$$\begin{pmatrix} x \\ \xi \\ \nu \end{pmatrix} \mapsto \begin{pmatrix} w_+ = \nu^{1/2} \cdot x + \nu^{-1/2} \cdot J^{-1}(\xi) \\ w_- = \nu^{1/2} \cdot x - \nu^{-1/2} \cdot J^{-1}(\xi) \\ \nu \end{pmatrix}$$

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- Φ preserves the Euclidean norm (up to an absolute const.)

$$
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\| & & \| \\
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where

$$L_0 = (\det(\mathrm{Id} + A^* A)/2)^{1/4} \cdot P \circ L' \circ P$$

with L' is pull-back by A restricted on $\mathbb{R}^2$ and P the projection

$$Pu(x) = c \int \exp(i \cdot \omega_0(x, y)/2 - |x - y|^2/4) u(y) dy$$

The last part of the proof

Finally we show

(1) $\|L_0 : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\| \leq 1.$

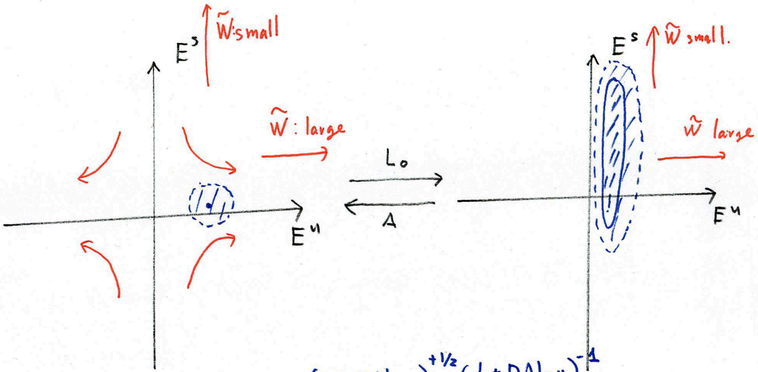
(2) $\|L_0 : L^2(\mathbb{R}^2, \widetilde{W}) \to L^2(\mathbb{R}^2, \widetilde{W})\| \leq \det(\mathrm{Id} + A^* A)^{-1/4} \sim \lambda^{-1/2}$

(1) can be proved by a rather general argument. To see (2), we have only to apply the crude estimate on the kernel of P:

$$|\exp(i\omega_0(x, y)/2 - |x - y|^2/4)| = \exp(-|x - y|^2/4)$$

and the definition of the weight function W.

In w_--plane



$$\| L_0 : L^2(\mathbb{R}^2, W) \circlearrowleft \| \lesssim (\det DA|_{E^u})^{+1/2} (\det DA|_{E^u})^{-1}$$
$$\sim \det DA|_{E^u})^{-1/2}$$