Mixing time-changes of parabolic flows

Corinna Ulcigrai
University of Bristol
(joint work with
Artur Avila and Giovanni Forni)

Corinaldo HDSS, June 2010

Parabolic flows

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems:* nearby orbits diverge exponentially
- *Parabolic dynamical systems:* nearby orbits diverge polynomially
- *Elliptic dynamical systems:* no divergence (or perhaps slower than polynomial)

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

Typical ergodic properties of parabolic dynamics: Unique ergodicity, polynomial speed of convergence of ergodic averages, polynomial decay of correlations, zero entropy, obstructions to the solutions of the chohomological equation.

We will be intererested in the presence of *mixing* in parabolic flows and their time-changes.

Parabolic flows

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems:* nearby orbits diverge exponentially
- *Parabolic dynamical systems:* nearby orbits diverge polynomially
- *Elliptic dynamical systems:* no divergence (or perhaps slower than polynomial)

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

Typical ergodic properties of parabolic dynamics: Unique ergodicity, polynomial speed of convergence of ergodic averages, polynomial decay of correlations, zero entropy, obstructions to the solutions of the chohomological equation.

We will be intererested in the presence of *mixing* in parabolic flows and their time-changes.

Parabolic flows

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems:* nearby orbits diverge exponentially
- *Parabolic dynamical systems:* nearby orbits diverge polynomially
- *Elliptic dynamical systems:* no divergence (or perhaps slower than polynomial)

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

Typical ergodic properties of parabolic dynamics: Unique ergodicity, polynomial speed of convergence of ergodic averages, polynomial decay of correlations, zero entropy, obstructions to the solutions of the chohomological equation.

We will be intererested in the presence of *mixing* in parabolic flows and their time-changes.

Parabolic flows

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems:* nearby orbits diverge exponentially
- *Parabolic dynamical systems:* nearby orbits diverge polynomially
- *Elliptic dynamical systems:* no divergence (or perhaps slower than polynomial)

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

Typical ergodic properties of parabolic dynamics: Unique ergodicity, polynomial speed of convergence of ergodic averages, polynomial decay of correlations, zero entropy, obstructions to the solutions of the chohomological equation.

We will be intererested in the presence of *mixing* in parabolic flows and their time-changes.

Parabolic flows

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems:* nearby orbits diverge exponentially
- *Parabolic dynamical systems:* nearby orbits diverge polynomially
- *Elliptic dynamical systems:* no divergence (or perhaps slower than polynomial)

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

Typical ergodic properties of parabolic dynamics: Unique ergodicity, polynomial speed of convergence of ergodic averages, polynomial decay of correlations, zero entropy, obstructions to the solutions of the chohomological equation.

We will be intererested in the presence of *mixing* in parabolic flows and their time-changes.

Parabolic flows

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems:* nearby orbits diverge exponentially
- *Parabolic dynamical systems:* nearby orbits diverge polynomially
- *Elliptic dynamical systems:* no divergence (or perhaps slower than polynomial)

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

Typical ergodic properties of parabolic dynamics: Unique ergodicity, polynomial speed of convergence of ergodic averages, polynomial decay of correlations, zero entropy, obstructions to the solutions of the chohomological equation.

We will be intererested in the presence of *mixing* in parabolic flows and their time-changes.

Parabolic flows

Dynamical systems can be roughly diveded into:

- *Hyperbolic dynamical systems:* nearby orbits diverge exponentially
- *Parabolic dynamical systems:* nearby orbits diverge polynomially
- *Elliptic dynamical systems:* no divergence (or perhaps slower than polynomial)

Examples of Parabolic flows:

- Horocycle flows on compact negatively curved manifolds;
- Area-preserving flows on surfaces of higher genus ($g \geq 2$);
- Nilflows on nilmanifolds (basic example: Heisenberg nilflows);

Typical ergodic properties of parabolic dynamics: Unique ergodicity, polynomial speed of convergence of ergodic averages, polynomial decay of correlations, zero entropy, obstructions to the solutions of the chohomological equation.

We will be intererested in the presence of *mixing* in parabolic flows and their time-changes.

The Heisenberg group

The 3-dimensional *Heisenberg group* N, up to isomorphisms, is the group of upper triangular unipotent matrices

$$[x, y, z] := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R}.$$

Definition

$N$is the unique connected, simply connected Lie group with 3-dimensional Lie algebra $\mathfrak{n}$ on two generators X, Y satisfying the Heisenberg commutation relations

$$[X, Y] = Z\,, \quad [X, Z] = [Y, Z] = 0\,.$$

A basis of the Lie algebra $\mathfrak{n}$ satisfying the Heisenberg commutations relations is given by the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Heisenberg nilmanifold

Definition

A compact *Heisenberg nilmanifold* is the quotient $M := \Gamma\backslash N$ of the Heisenberg group over a co-compact lattice $\Gamma < N$.

It is well-known that there exists a positive integer $E \in \mathbb{N}$ such that, up to an automorphism of N, the lattice Γ coincide with the lattice

$$\Gamma := \left\{ \begin{pmatrix} 1 & x & z/E \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \,:\, x, y, z \in \mathbb{Z} \right\}. \qquad (\text{take e.g. } E = 1)$$

The group N acts on the right transitively on M by right multiplication:

$$R_g(x) := x\, g, \quad x \in M,\ g \in N.$$

Definition

Heisenberg nilflows are the flows obtained by the restriction of this right action to the one-parameter subgroups on N.

Any Heisenberg nilmanifold M has a natural probability measure μ locally given by the Haar measure of N; μ is invariant under all nilflows on M.

The Heisenberg nilmanifold

Definition

A compact *Heisenberg nilmanifold* is the quotient $M := \Gamma\backslash N$ of the Heisenberg group over a co-compact lattice $\Gamma < N$.

It is well-known that there exists a positive integer $E \in \mathbb{N}$ such that, up to an automorphism of N, the lattice Γ coincide with the lattice

$$\Gamma := \left\{ \begin{pmatrix} 1 & x & z/E \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \ x, y, z \in \mathbb{Z} \right\}. \qquad (\text{take e.g. } \ E = 1)$$

The group N acts on the right transitively on M by right multiplication:

$$R_g(x) := x\, g, \quad x \in M,\ g \in N.$$

Definition

Heisenberg nilflows are the flows obtained by the restriction of this right action to the one-parameter subgroups on N.

Any Heisenberg nilmanifold M has a natural probability measure μ locally given by the Haar measure of N; μ is invariant under all nilflows on M.

Skew shifts as return maps of Heisenberg nilflows

Lemma

Any uniquely ergodic Heisenberg nilflow admits a cross section Σ isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that the Poincaré first return map to Σ is a linear skew shift over a circle rotation, i.e.

$$f(x,y) := (x+\alpha, y+x+\beta)\,, \quad \textit{for all } (x,y) \in \mathbb{T}^2, \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

Proof.

Let $\Sigma \subset M$ be the smooth surface defined by:

$$\Sigma := \{\Gamma \exp(xX + zZ) : \ (x,z) \in \mathbb{R}^2\}\,.$$

The map $(x,z) \mapsto \Gamma \exp(xX + zZ)$ gives an isomorphism with $\mathbb{T}^2$ since $< X, Z >$ is an abelian ideal of $\mathfrak{n}$.
If $\phi^W = \{\phi^W_t\}_{t\in\mathbb{R}}$ is the uniquely ergodic Heisenberg nilflow generated by $W := w_x X + w_y Y + w_z Z$, the first return map to Σ is:

$$(x,z) \mapsto (x + \frac{w_x}{w_y}, z + x + \frac{w_z}{w_y} + \frac{w_x}{2w_y})\,, \quad (x,z) \in \mathbb{T}^2\,.$$

Skew shifts as return maps of Heisenberg nilflows

Lemma

Any uniquely ergodic Heisenberg nilflow admits a cross section Σ isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that the Poincaré first return map to Σ is a linear skew shift over a circle rotation, i.e.

$$f(x,y) := (x+\alpha, y+x+\beta)\,, \quad \textit{for all } (x,y) \in \mathbb{T}^2, \quad \text{where}\, \alpha, \beta \in \mathbb{R}.$$

Proof.

Let $\Sigma \subset M$ be the smooth surface defined by:

$$\Sigma := \{\Gamma \exp(xX + zZ) : \ (x,z) \in \mathbb{R}^2\}\,.$$

The map $(x,z) \mapsto \Gamma \exp(xX + zZ)$ gives an isomorphism with $\mathbb{T}^2$ since $< X, Z >$ is an abelian ideal of $\mathfrak{n}$.

If $\phi^W = \{\phi^W_t\}_{t\in\mathbb{R}}$ is the uniquely ergodic Heisenberg nilflow generated by $W := w_x X + w_y Y + w_z Z$, the first return map to Σ is:

$$(x,z) \mapsto (x + \frac{w_x}{w_y}, z + x + \frac{w_z}{w_y} + \frac{w_x}{2w_y})\,, \quad (x,z) \in \mathbb{T}^2\,.$$

Skew shifts as return maps of Heisenberg nilflows

Lemma

Any uniquely ergodic Heisenberg nilflow admits a cross section Σ isomorphic to $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that the Poincaré first return map to Σ is a linear skew shift over a circle rotation, i.e.

$$f(x,y) := (x+\alpha, y+x+\beta)\,, \quad \text{for all } (x,y) \in \mathbb{T}^2, \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

Proof.

Let $\Sigma \subset M$ be the smooth surface defined by:

$$\Sigma := \{\Gamma \exp(xX + zZ) : \ (x,z) \in \mathbb{R}^2\}\,.$$

The map $(x,z) \mapsto \Gamma \exp(xX + zZ)$ gives an isomorphism with $\mathbb{T}^2$ since $< X, Z >$ is an abelian ideal of $\mathfrak{n}$.
If $\phi^W = \{\phi^W_t\}_{t\in\mathbb{R}}$ is the uniquely ergodic Heisenberg nilflow generated by $W := w_x X + w_y Y + w_z Z$, the first return map to Σ is:

$$(x,z) \mapsto (x + \frac{w_x}{w_y}, z + x + \frac{w_z}{w_y} + \frac{w_x}{2w_y})\,, \quad (x,z) \in \mathbb{T}^2\,.$$

Special flow representation of Heisenberg nilflows

Moreover one can compute the first return time function Φ of the flow ϕ^W to the transverse section Σ. It is *constant* and given by $\Phi \equiv 1/w_y$.

Thus:

Lemma

any (uniquely ergodic) Heisenberg nilflow ϕ^W is smoothly isomorphic to a special flow over a linear skew-shift of the form $(x, y) \mapsto (x + \alpha, y + x + \beta)$ with constant roof function Φ.

Recall that:

The *special flow* $f^\Phi = \{f_t^\Phi\}_{t\in\mathbb{R}}$
over the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$
under the roof function $\Phi\colon \mathbb{T}^2 \to \mathbb{R}^+$

is the quotient of the unit speed vertical flow on $\times\mathbb{R}$ $\dot{z} = 1$ on the phase space $\{((x, y), z) \in \Sigma \times \mathbb{R}\}$ with respect to the equivalence relation $\sim_\Phi$ defined by $((x, y), \Phi(x, y) + z) \sim_\Phi (f(x, y), z)$.

Special flow representation of Heisenberg nilflows

Moreover one can compute the first return time function Φ of the flow ϕ^W to the transverse section Σ. It is *constant* and given by $\Phi \equiv 1/w_y$.

Thus:

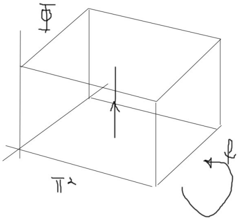
Lemma

any (uniquely ergodic) Heisenberg nilflow ϕ^W is smoothly isomorphic to a special flow over a linear skew-shift of the form $(x, y) \mapsto (x + \alpha, y + x + \beta)$ with constant roof function Φ.

Recall that:

The *special flow* $f^\Phi = \{f_t^\Phi\}_{t\in\mathbb{R}}$
over the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$
under the roof function $\Phi\colon \mathbb{T}^2 \to \mathbb{R}^+$

is the quotient of the unit speed vertical flow on $\times\mathbb{R}$ $\dot{z} = 1$ on the phase space $\{((x, y), z) \in \Sigma \times \mathbb{R}\}$ with respect to the equivalence relation $\sim_\Phi$ defined by $((x, y), \Phi(x, y) + z) \sim_\Phi (f(x, y), z)$.



Special flow representation of Heisenberg nilflows

Moreover one can compute the first return time function Φ of the flow ϕ^W to the transverse section Σ. It is *constant* and given by $\Phi \equiv 1/w_y$.

Thus:

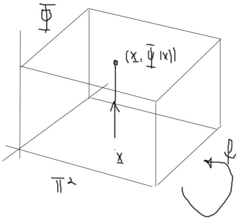
Lemma

any (uniquely ergodic) Heisenberg nilflow ϕ^W is smoothly isomorphic to a special flow over a linear skew-shift of the form $(x, y) \mapsto (x + \alpha, y + x + \beta)$ with constant roof function Φ.

Recall that:

The *special flow* $f^\Phi = \{f_t^\Phi\}_{t\in\mathbb{R}}$
over the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$
under the roof function $\Phi\colon \mathbb{T}^2 \to \mathbb{R}^+$

is the quotient of the unit speed vertical flow on $\times\mathbb{R}$ $\dot{z} = 1$ on the phase space $\{((x, y), z) \in \Sigma \times \mathbb{R}\}$ with respect to the equivalence relation $\sim_\Phi$ defined by $((x, y), \Phi(x, y) + z) \sim_\Phi (f(x, y), z)$.



Special flow representation of Heisenberg nilflows

Moreover one can compute the first return time function Φ of the flow ϕ^W to the transverse section Σ. It is *constant* and given by $\Phi \equiv 1/w_y$.

Thus:

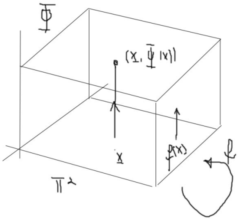
Lemma

any (uniquely ergodic) Heisenberg nilflow ϕ^W is smoothly isomorphic to a special flow over a linear skew-shift of the form $(x, y) \mapsto (x + \alpha, y + x + \beta)$ with constant roof function Φ.

Recall that:

The *special flow* $f^\Phi = \{f^\Phi_t\}_{t\in\mathbb{R}}$
over the map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$
under the roof function $\Phi\colon \mathbb{T}^2 \to \mathbb{R}^+$

is the quotient of the unit speed vertical flow on $\times\mathbb{R}$ $\dot{z} = 1$ on the phase space $\{((x, y), z) \in \Sigma \times \mathbb{R}\}$ with respect to the equivalence relation $\sim_\Phi$ defined by $((x, y), \Phi(x, y) + z) \sim_\Phi (f(x, y), z)$.



Special flow representation of Heisenberg nilflows

Moreover one can compute the first return time function Φ of the flow ϕ^W to the transverse section Σ. It is *constant* and given by $\Phi \equiv 1/w_y$.

Thus:

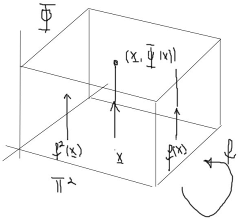
Lemma

any (uniquely ergodic) Heisenberg nilflow ϕ^W is smoothly isomorphic to a special flow over a linear skew-shift of the form $(x, y) \mapsto (x + \alpha, y + x + \beta)$ *with constant roof function* Φ.

Recall that:

The *special flow* $f^\Phi = \{f_t^\Phi\}_{t\in\mathbb{R}}$
over the map $f \colon \mathbb{T}^2 \to \mathbb{T}^2$
under the roof function $\Phi \colon \mathbb{T}^2 \to \mathbb{R}^+$

is the quotient of the unit speed vertical flow on $\times\mathbb{R}$ $\dot{z} = 1$ on the phase space $\{((x, y), z) \in \Sigma \times \mathbb{R}\}$ with respect to the equivalence relation $\sim_\Phi$ defined by $((x, y), \Phi(x, y) + z) \sim_\Phi (f(x, y), z)$.



Mixing in parabolic flows

Recall that a measure preserving flow $\{h_t\}_{t\in\mathbb{R}}$ is *mixing* if for all measurable sets A, B we have

$$\mu(A \cap h_t(B)) \xrightarrow{t\to\infty} \mu(A)\mu(B). \tag{1}$$

Naive question: are parabolic flows mixing? mixing with polynomial decay of correlations?

In the previous Examples:

- The Horocycle flows is mixing and mixing of all orders (Marcus)
- Area preserving flows on surfaces: mixing depends on the parametrization and on the type of singularities (see later).
- Nilflows on nilmanifolds: never (weak) mixing.

General philosophy: If a parabolic flow is not mixing, can one reparametrize it (find a time-change) such that it becomes mixing? mixing with polynomial decay of correlations?

Mixing in parabolic flows

Recall that a measure preserving flow $\{h_t\}_{t\in\mathbb{R}}$ is *mixing* if for all measurable sets A, B we have

$$\mu(A \cap h_t(B)) \xrightarrow{t\to\infty} \mu(A)\mu(B). \qquad (1)$$

Naive question: are parabolic flows mixing? mixing with polynomial decay of correlations?

In the previous Examples:

- The Horocycle flows is mixing and mixing of all orders (Marcus)
- Area preserving flows on surfaces: mixing depends on the parametrization and on the type of singularities (see later).
- Nilflows on nilmanifolds: never (weak) mixing.

General philosophy: If a parabolic flow is not mixing, can one reparametrize it (find a time-change) such that it becomes mixing? mixing with polynomial decay of correlations?

Mixing in parabolic flows

Recall that a measure preserving flow $\{h_t\}_{t\in\mathbb{R}}$ is *mixing* if for all measurable sets A, B we have

$$\mu(A \cap h_t(B)) \xrightarrow{t\to\infty} \mu(A)\mu(B). \tag{1}$$

Naive question: are parabolic flows mixing? mixing with polynomial decay of correlations?

In the previous Examples:

- The Horocycle flows is mixing and mixing of all orders (Marcus)
- Area preserving flows on surfaces: mixing depends on the parametrization and on the type of singularities (see later).
- Nilflows on nilmanifolds: never (weak) mixing.

General philosophy: If a parabolic flow is not mixing, can one reparametrize it (find a time-change) such that it becomes mixing? mixing with polynomial decay of correlations?

Mixing in parabolic flows

Recall that a measure preserving flow $\{h_t\}_{t\in\mathbb{R}}$ is *mixing* if for all measurable sets A, B we have

$$\mu(A \cap h_t(B)) \xrightarrow{t\to\infty} \mu(A)\mu(B). \tag{1}$$

Naive question: are parabolic flows mixing? mixing with polynomial decay of correlations?

In the previous Examples:

- The Horocycle flows is mixing and mixing of all orders (Marcus)
- Area preserving flows on surfaces: mixing depends on the parametrization and on the type of singularities (see later).
- Nilflows on nilmanifolds: never (weak) mixing.

General philosophy: If a parabolic flow is not mixing, can one reparametrize it (find a time-change) such that it becomes mixing? mixing with polynomial decay of correlations?

Mixing in parabolic flows

Recall that a measure preserving flow $\{h_t\}_{t\in\mathbb{R}}$ is *mixing* if for all measurable sets A, B we have

$$\mu(A \cap h_t(B)) \xrightarrow{t\to\infty} \mu(A)\mu(B). \tag{1}$$

Naive question: are parabolic flows mixing? mixing with polynomial decay of correlations?

In the previous Examples:

- The Horocycle flows is mixing and mixing of all orders (Marcus)
- Area preserving flows on surfaces: mixing depends on the parametrization and on the type of singularities (see later).
- Nilflows on nilmanifolds: never (weak) mixing.

General philosophy: If a parabolic flow is not mixing, can one reparametrize it (find a time-change) such that it becomes mixing? mixing with polynomial decay of correlations?

Mixing in parabolic flows

Recall that a measure preserving flow $\{h_t\}_{t\in\mathbb{R}}$ is *mixing* if for all measurable sets A, B we have

$$\mu(A \cap h_t(B)) \xrightarrow{t\to\infty} \mu(A)\mu(B). \tag{1}$$

Naive question: are parabolic flows mixing? mixing with polynomial decay of correlations?

In the previous Examples:

- The Horocycle flows is mixing and mixing of all orders (Marcus)
- Area preserving flows on surfaces: mixing depends on the parametrization and on the type of singularities (see later).
- Nilflows on nilmanifolds: never (weak) mixing.

General philosophy: If a parabolic flow is not mixing, can one reparametrize it (find a time-change) such that it becomes mixing? mixing with polynomial decay of correlations?

Time-changes

Intuition: if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{h_t\}_{t\in\mathbb{R}}$, the trajectories of $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ are the same than $\{h_t\}_{t\in\mathbb{R}}$ but the speed is different.

Definition

A flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *time-change* of a flow $\{h_t\}_{t\in\mathbb{R}}$ on X (or a *reparametrization*) if there exists $\tau : X \times \mathbb{R} \to \mathbb{R}$ s.t.

$$\forall x \in X, t \in \mathbb{R}, \qquad \widetilde{h}_t(x) = h_{\tau(x,t)}(x).$$

Since $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a flow, τ is an *additive cocycle*, i.e.

$$\tau(x, s+t) = \tau(\widetilde{h}_s(x), t) + \tau(x, s), \qquad \text{for all } x \in X, \; s, t \in \mathbb{R}.$$

If X is a manifold and $\{h_t\}_{t\in\mathbb{R}}$ is a smooth flow, we will say that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *smooth reparametrization* if the cocycle τ is a smooth function. In this case we also have

$$\frac{\partial \widetilde{h}_t}{\partial t}(x,0) = \alpha(x)\frac{\partial h_t}{\partial t}(x,0)$$

Time-changes

Intuition: if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{h_t\}_{t\in\mathbb{R}}$, the trajectories of $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ are the same than $\{h_t\}_{t\in\mathbb{R}}$ but the speed is different.

Definition

A flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *time-change* of a flow $\{h_t\}_{t\in\mathbb{R}}$ on X (or a *reparametrization*) if there exists $\tau : X \times \mathbb{R} \to \mathbb{R}$ s.t.

$$\forall x \in X, t \in \mathbb{R}, \qquad \widetilde{h}_t(x) = h_{\tau(x,t)}(x).$$

Since $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a flow, τ is an *additive cocycle*, i.e.

$$\tau(x, s+t) = \tau(\widetilde{h}_s(x), t) + \tau(x, s), \quad \text{for all } x \in X, \ s, t \in \mathbb{R}.$$

If X is a manifold and $\{h_t\}_{t\in\mathbb{R}}$ is a smooth flow, we will say that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *smooth reparametrization* if the cocycle τ is a smooth function. In this case we also have

$$\frac{\partial \widetilde{h}_t}{\partial t}(x,0) = \alpha(x)\frac{\partial h_t}{\partial t}(x,0)$$

Time-changes

Intuition: if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{h_t\}_{t\in\mathbb{R}}$, the trajectories of $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ are the same than $\{h_t\}_{t\in\mathbb{R}}$ but the speed is different.

Definition

A flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *time-change* of a flow $\{h_t\}_{t\in\mathbb{R}}$ on X (or a *reparametrization*) if there exists $\tau : X\times\mathbb{R}\to\mathbb{R}$ s.t.

$$\forall x\in X, t\in\mathbb{R}, \qquad \widetilde{h}_t(x) = h_{\tau(x,t)}(x).$$

Since $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a flow, τ is an *additive cocycle*, i.e.

$$\tau(x, s+t) = \tau(\widetilde{h}_s(x), t) + \tau(x,s)\,, \qquad \text{for all } x\in X\,,\ s,t\in\mathbb{R}\,.$$

If X is a manifold and $\{h_t\}_{t\in\mathbb{R}}$ is a smooth flow, we will say that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *smooth reparametrization* if the cocycle τ is a smooth function. In this case we also have

$$\frac{\partial \widetilde{h}_t}{\partial t}(x,0) = \alpha(x)\frac{\partial h_t}{\partial t}(x,0)$$

Time-changes

Intuition: if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{h_t\}_{t\in\mathbb{R}}$, the trajectories of $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ are the same than $\{h_t\}_{t\in\mathbb{R}}$ but the speed is different.

Definition

A flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *time-change* of a flow $\{h_t\}_{t\in\mathbb{R}}$ on X (or a *reparametrization*) if there exists $\tau : X \times \mathbb{R} \to \mathbb{R}$ s.t.

$$\forall x \in X, t \in \mathbb{R}, \qquad \widetilde{h}_t(x) = h_{\tau(x,t)}(x).$$

Since $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a flow, τ is an *additive cocycle*, i.e.

$$\tau(x, s+t) = \tau(\widetilde{h}_s(x), t) + \tau(x, s)\,, \qquad \text{for all } x \in X\,,\ s, t \in \mathbb{R}\,.$$

If X is a manifold and $\{h_t\}_{t\in\mathbb{R}}$ is a smooth flow, we will say that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a *smooth reparametrization* if the cocycle τ is a smooth function. In this case we also have

$$\frac{\partial \widetilde{h}_t}{\partial t}(x,0) = \alpha(x)\frac{\partial h_t}{\partial t}(x,0)$$

Horocycle Flow

- The horocycle flow on compact negatively curved manifolds is mixing and mixing of all orders (Marcus)
- decay of correlations of smooth functions is *polynomial* in time (Ratner);

One can ask the converse question: does mixing persist under time-changes?

- Kuschnirenko has proved that if the time-change is sufficiently small (in the C^1 topology), the time-change is still mixing.

Open Questions: Does this result (persistence of mixing) extends to all smooth time-changes?

Horocycle Flow

- The horocycle flow on compact negatively curved manifolds is mixing and mixing of all orders (Marcus)
- decay of correlations of smooth functions is *polynomial* in time (Ratner);

One can ask the converse question: does mixing persist under time-changes?

- Kuschnirenko has proved that if the time-change is sufficiently small (in the C^1 topology), the time-change is still mixing.

Open Questions: Does this result (persistence of mixing) extends to all smooth time-changes?

Horocycle Flow

- The horocycle flow on compact negatively curved manifolds is mixing and mixing of all orders (Marcus)
- decay of correlations of smooth functions is *polynomial* in time (Ratner);

One can ask the converse question: does mixing persist under time-changes?

- Kuschnirenko has proved that if the time-change is sufficiently small (in the C^1 topology), the time-change is still mixing.

Open Questions: Does this result (persistence of mixing) extends to all smooth time-changes?

Horocycle Flow

- The horocycle flow on compact negatively curved manifolds is mixing and mixing of all orders (Marcus)
- decay of correlations of smooth functions is *polynomial* in time (Ratner);

One can ask the converse question: does mixing persist under time-changes?

- Kuschnirenko has proved that if the time-change is sufficiently small (in the C^1 topology), the time-change is still mixing.

Open Questions: Does this result (persistence of mixing) extends to all smooth time-changes?

Area-preserving flows on surfaces

Mixing depends on the parametrization:

- Translation surface flows (arise from billiards in rational polygons);
- Locally Hamiltonian flows on surfaces (Novikov);

Translation surfaces can be obtained glueing opposite parallel sides of polygons. The linear unit speed flow in the polygon quotient to a flow with singularities on the surface (the translation surface directional flow).

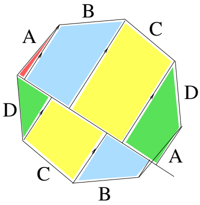
- The translation surface flow (linear flow with unit-speed) is never mixing. Smooth time-changes are also not mixing (both proven by Katok , 80s).

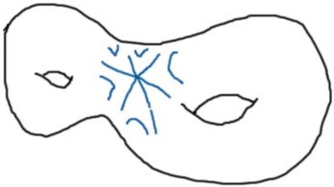
Area-preserving flows on surfaces

Mixing depends on the parametrization:

- Translation surface flows (arise from billiards in rational polygons);
- Locally Hamiltonian flows on surfaces (Novikov);

Translation surfaces flows:





Translation surfaces can be obtained glueing opposite parallel sides of polygons. The linear unit speed flow in the polygon quotient to a flow with singularities on the surface (the translation surface directional flow).

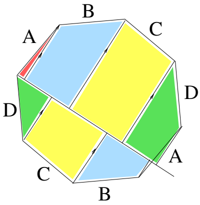
- The translation surface flow (linear flow with unit-speed) is never mixing. Smooth time-changes are also not mixing (both proven by Katok , 80s).

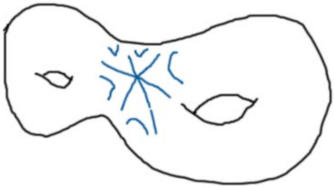
Area-preserving flows on surfaces

Mixing depends on the parametrization:

- Translation surface flows (arise from billiards in rational polygons);
- Locally Hamiltonian flows on surfaces (Novikov);

Translation surfaces flows:



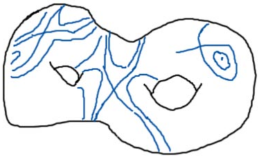


Translation surfaces can be obtained glueing opposite parallel sides of polygons. The linear unit speed flow in the polygon quotient to a flow with singularities on the surface (the translation surface directional flow).

- The translation surface flow (linear flow with unit-speed) is never mixing. Smooth time-changes are also not mixing (both proven by Katok , 80s).

Area-preserving flows on surfaces

Locally Hamiltonian flows:



Locally solutions to

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

dH closed 1-form

Minimal components are time-changes of translation surface flows.
Mixing depends delicately on singularities type:

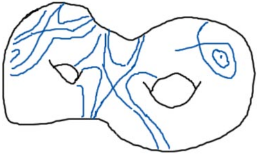
If there is a degenerate saddle (non typical) the flow is mixing (Kochergin) (polynomially for $g = 1$, Fayad)

If there are saddle loops, minimal components are typically mixing (U'07) (for $g = 1$, Sinai-Khanin)

Typical minimal flows with only simple saddles are NOT mixing (but weak mixing) U'09
($g = 1$ Kochergin, Fraczek-Lemanczyk, $g = 2$ Scheglov)

Area-preserving flows on surfaces

Locally Hamiltonian flows:



Locally solutions to

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

dH closed 1-form

Minimal components are time-changes of translation surface flows.
Mixing depends delicately on singularities type:

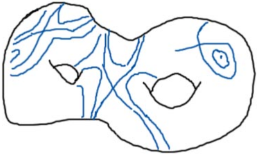
If there is a degenerate saddle (non typical) the flow is mixing (Kochergin) (polynomially for $g = 1$, Fayad)

If there are saddle loops, minimal components are typically mixing (U'07) (for $g = 1$, Sinai-Khanin)

Typical minimal flows with only simple saddles are NOT mixing (but weak mixing) U'09
($g = 1$ Kochergin, Fraczek-Lemanczyk, $g = 2$ Scheglov)

Area-preserving flows on surfaces

Locally Hamiltonian flows:



Locally solutions to

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

dH closed 1-form

Minimal components are time-changes of translation surface flows.
Mixing depends delicately on singularities type:

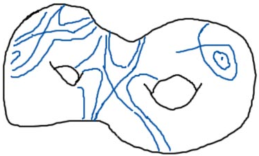
If there is a degenerate saddle (non typical) the flow is mixing (Kochergin) (polynomially for $g = 1$, Fayad)

If there are saddle loops, minimal components are typically mixing (U'07) (for $g = 1$, Sinai-Khanin)

Typical minimal flows with only simple saddles are NOT mixing (but weak mixing) U'09
($g = 1$ Kochergin, Fraczek-Lemanczyk, $g = 2$ Scheglov)

Area-preserving flows on surfaces

Locally Hamiltonian flows:



Locally solutions to

$$\dot{x} = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{\partial H}{\partial x}$$

dH closed 1-form

Minimal components are time-changes of translation surface flows.
Mixing depends delicately on singularities type:

If there is a degenerate saddle (non typical) the flow is mixing (Kochergin) (polynomially for $g = 1$, Fayad)

If there are saddle loops, minimal components are typically mixing (U'07) (for $g = 1$, Sinai-Khanin)

Typical minimal flows with only simple saddles are NOT mixing (but weak mixing) U'09
($g = 1$ Kochergin, Fraczek-Lemanczyk, $g = 2$ Scheglov)

Dictionary between time-changes and special flows

Time-changes	vs	Special flows
original flow $\{h_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under Φ
time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under new roof $\widetilde{\Phi}$
smooth time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	*smooth* new roof $\widetilde{\Phi}$
trivial time change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ ($\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ conjugated to $\{h_t\}_{t\in\mathbb{R}}$)	$\leftrightarrow$	*cohomologous* roof $\widetilde{\Phi}$ $\exists h$ s.t. $\widetilde{\Phi} = \Phi + h\circ f - h$
smoothly trivial	$\leftrightarrow$	h smooth

Dictionary between time-changes and special flows

Time-changes	vs	Special flows
original flow $\{h_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under Φ
time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under new roof $\widetilde{\Phi}$
smooth time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	*smooth* new roof $\widetilde{\Phi}$
trivial time change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ ($\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ conjugated to $\{h_t\}_{t\in\mathbb{R}}$)	$\leftrightarrow$	*cohomologous* roof $\widetilde{\Phi}$ $\exists h$ s.t. $\widetilde{\Phi} = \Phi + h\circ f - h$
smoothly trivial	$\leftrightarrow$	h smooth

Dictionary between time-changes and special flows

Time-changes	vs	Special flows
original flow $\{h_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under Φ
time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under new roof $\widetilde{\Phi}$
smooth time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	*smooth* new roof $\widetilde{\Phi}$
trivial time change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ ($\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ conjugated to $\{h_t\}_{t\in\mathbb{R}}$)	$\leftrightarrow$	*cohomologous* roof $\widetilde{\Phi}$ $\exists h$ s.t. $\widetilde{\Phi} = \Phi + h \circ f - h$
smoothly trivial	$\leftrightarrow$	h smooth

Dictionary between time-changes and special flows

Time-changes	vs	Special flows
original flow $\{h_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under Φ
time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under new roof $\widetilde{\Phi}$
smooth time-change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	*smooth* new roof $\widetilde{\Phi}$
trivial time change $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ ($\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ conjugated to $\{h_t\}_{t\in\mathbb{R}}$)	$\leftrightarrow$	*cohomologous* roof $\widetilde{\Phi}$ $\exists h$ s.t. $\widetilde{\Phi} = \Phi + h\circ f - h$
smoothly trivial	$\leftrightarrow$	h smooth

Dictionary between time-changes and special flows

Time-changes	vs	Special flows
original flow $\{h_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under Φ
time-change $\{\widetilde{h_t}\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under new roof $\widetilde{\Phi}$
smooth time-change $\{\widetilde{h_t}\}_{t\in\mathbb{R}}$	$\leftrightarrow$	*smooth* new roof $\widetilde{\Phi}$
trivial time change $\{\widetilde{h_t}\}_{t\in\mathbb{R}}$ ($\{\widetilde{h_t}\}_{t\in\mathbb{R}}$ conjugated to $\{h_t\}_{t\in\mathbb{R}}$)	$\leftrightarrow$	*cohomologous* roof $\widetilde{\Phi}$ $\exists h$ s.t. $\widetilde{\Phi} = \Phi + h \circ f - h$
smoothly trivial	$\leftrightarrow$	h smooth

Dictionary between time-changes and special flows

Time-changes	vs	Special flows
original flow $\{h_t\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under Φ
time-change $\{\widetilde{h_t}\}_{t\in\mathbb{R}}$	$\leftrightarrow$	special flow under new roof $\widetilde{\Phi}$
smooth time-change $\{\widetilde{h_t}\}_{t\in\mathbb{R}}$	$\leftrightarrow$	*smooth* new roof $\widetilde{\Phi}$
trivial time change $\{\widetilde{h_t}\}_{t\in\mathbb{R}}$ ($\{\widetilde{h_t}\}_{t\in\mathbb{R}}$ conjugated to $\{h_t\}_{t\in\mathbb{R}}$)	$\leftrightarrow$	*cohomologous* roof $\widetilde{\Phi}$ $\exists h$ s.t. $\widetilde{\Phi} = \Phi + h\circ f - h$
smoothly trivial	$\leftrightarrow$	h smooth

Mixing time-changes for Heisenberg niflows

Assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Thus f is uniquely ergodic (equivalenty assume that the Heisenberg nilflow is uniquely ergodic).

Theorem (AFU)

There exist a dense subspace $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ (roof functions) and a subspace $\mathcal{T}_f \subset \mathcal{R}$ of countable codimension (trivial roofs) such that if we set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$ (mixing roofs), for any positive roof function Φ belonging to $\mathcal{M}_f$ the special flow f^Φ is mixing.

More precisely:

Corollary (AFU)

For any positive function $\Phi \in \mathcal{R}$ the following properties are equivalent:

1. *the roof function $\Phi \in \mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$;*
2. *the special flow f^Φ is not smoothly trivial;*
3. *the special flow f^Φ is weak mixing;*
4. *the special flow f^Φ is mixing.*

The Theorem and the Corollary can be rephrased for time-changes of Heisenberg nilflows using the dictionary.

Mixing time-changes for Heisenberg niflows

Assume $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Thus f is uniquely ergodic (equivalenty assume that the Heisenberg nilflow is uniquely ergodic).

Theorem (AFU)

There exist a dense subspace $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ (roof functions) and a subspace $\mathcal{T}_f \subset \mathcal{R}$ of countable codimension (trivial roofs) such that if we set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$ (mixing roofs), for any positive roof function Φ belonging to $\mathcal{M}_f$ the special flow f^Φ is mixing.

More precisely:

Corollary (AFU)

For any positive function $\Phi \in \mathcal{R}$ the following properties are equivalent:

1. *the roof function $\Phi \in \mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$;*
2. *the special flow f^Φ is not smoothly trivial;*
3. *the special flow f^Φ is weak mixing;*
4. *the special flow f^Φ is mixing.*

The Theorem and the Corollary can be rephrased for time-changes of Heisenberg nilflows using the dictionary.

Mixing time-changes for Heisenberg niflows

Assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Thus f is uniquely ergodic (equivalenty assume that the Heisenberg nilflow is uniquely ergodic).

Theorem (AFU)

There exist a dense subspace $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ (roof functions) and a subspace $\mathcal{T}_f \subset \mathcal{R}$ of countable codimension (trivial roofs) such that if we set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$ (mixing roofs), for any positive roof function Φ belonging to $\mathcal{M}_f$ the special flow f^Φ is mixing.

More precisely:

Corollary (AFU)

For any positive function $\Phi \in \mathcal{R}$ the following properties are equivalent:

1. *the roof function $\Phi \in \mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$;*
2. *the special flow f^Φ is not smoothly trivial;*
3. *the special flow f^Φ is weak mixing;*
4. *the special flow f^Φ is mixing.*

The Theorem and the Corollary can be rephrased for time-changes of Heisenberg nilflows using the dictionary.

Mixing time-changes for Heisenberg niflows

Assume $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Thus f is uniquely ergodic (equivalenty assume that the Heisenberg nilflow is uniquely ergodic).

Theorem (AFU)

There exist a dense subspace $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ (roof functions) and a subspace $\mathcal{T}_f \subset \mathcal{R}$ of countable codimension (trivial roofs) such that if we set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$ (mixing roofs), for any positive roof function Φ belonging to $\mathcal{M}_f$ the special flow f^{Φ} is mixing.

More precisely:

Corollary (AFU)

For any positive function $\Phi \in \mathcal{R}$ the following properties are equivalent:

1. *the roof function $\Phi \in \mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$;*
2. *the special flow f^{Φ} is not smoothly trivial;*
3. *the special flow f^{Φ} is weak mixing;*
4. *the special flow f^{Φ} is mixing.*

The Theorem and the Corollary can be rephrased for time-changes of Heisenberg nilflows using the dictionary.

Mixing time-changes for Heisenberg niflows

Assume $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Thus f is uniquely ergodic (equivalenty assume that the Heisenberg nilflow is uniquely ergodic).

Theorem (AFU)

There exist a dense subspace $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ (roof functions) and a subspace $\mathcal{T}_f \subset \mathcal{R}$ of countable codimension (trivial roofs) such that if we set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$ (mixing roofs), for any positive roof function Φ belonging to $\mathcal{M}_f$ the special flow f^Φ is mixing.

More precisely:

Corollary (AFU)

For any positive function $\Phi \in \mathcal{R}$ the following properties are equivalent:

1. *the roof function $\Phi \in \mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$;*
2. *the special flow f^Φ is not smoothly trivial;*
3. *the special flow f^Φ is weak mixing;*
4. *the special flow f^Φ is mixing.*

The Theorem and the Corollary can be rephrased for time-changes of Heisenberg nilflows using the dictionary.

Mixing time-changes for Heisenberg niflows

Assume $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Thus f is uniquely ergodic (equivalenty assume that the Heisenberg nilflow is uniquely ergodic).

Theorem (AFU)

There exist a dense subspace $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ (roof functions) and a subspace $\mathcal{T}_f \subset \mathcal{R}$ of countable codimension (trivial roofs) such that if we set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$ (mixing roofs), for any positive roof function Φ belonging to $\mathcal{M}_f$ the special flow f^Φ is mixing.

More precisely:

Corollary (AFU)

For any positive function $\Phi \in \mathcal{R}$ the following properties are equivalent:

1. *the roof function $\Phi \in \mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$;*
2. *the special flow f^Φ is not smoothly trivial;*
3. *the special flow f^Φ is weak mixing;*
4. *the special flow f^Φ is mixing.*

The Theorem and the Corollary can be rephrased for time-changes of Heisenberg nilflows using the dictionary.

Mixing time-changes for Heisenberg niflows

Assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Thus f is uniquely ergodic (equivalenty assume that the Heisenberg nilflow is uniquely ergodic).

Theorem (AFU)

There exist a dense subspace $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ (roof functions) and a subspace $\mathcal{T}_f \subset \mathcal{R}$ of countable codimension (trivial roofs) such that if we set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$ (mixing roofs), for any positive roof function Φ belonging to $\mathcal{M}_f$ the special flow f^Φ is mixing.

More precisely:

Corollary (AFU)

For any positive function $\Phi \in \mathcal{R}$ the following properties are equivalent:

1. *the roof function $\Phi \in \mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$;*
2. *the special flow f^Φ is not smoothly trivial;*
3. *the special flow f^Φ is weak mixing;*
4. *the special flow f^Φ is mixing.*

The Theorem and the Corollary can be rephrased for time-changes of Heisenberg nilflows using the dictionary.

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3,\ k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j + \alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3,\ k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j+\alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).

   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3$, $k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j+\alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3$, $k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j+\alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x,y) = \sin(2\pi y) + 2$;
   - $\Phi(x,y) = \cos(2\pi(kx+y)) + \sin(2\pi lx) + 3$, $k,l \in \mathbb{Z}$;
   - $\Phi(x,y) = \operatorname{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j + \alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3,\ k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j+\alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.
   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3,\ k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j+\alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3$, $k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j+\alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3$, $k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j+\alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x,y) = \sin(2\pi y) + 2$;
   - $\Phi(x,y) = \cos(2\pi(kx+y)) + \sin(2\pi lx) + 3$, $k, l \in \mathbb{Z}$;
   - $\Phi(x,y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i (jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i (\beta j + \alpha \binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Remarks and questions on mixing time-changes:

Remarks:

1. Weak mixing is equivalent to mixing (in the class $\mathcal{R}$);
2. The generic subset $\mathcal{M}_f$ in the main Theorem is *concretely* described (in terms of invariant distributions).
   Tt is possible to check *explicitely* if a given smooth roof function given in terms of a Fourier expansion belongs to $\mathcal{M}_f$ and to give concrete examples of mixing reparametrizations.

   Examples.

   - $\Phi(x, y) = \sin(2\pi y) + 2$;
   - $\Phi(x, y) = \cos(2\pi(kx + y)) + \sin(2\pi lx) + 3,\ k, l \in \mathbb{Z}$;
   - $\Phi(x, y) = \mathrm{Re} \sum_{j\in\mathbb{Z}} a_j e^{2\pi i(jx+y)} + c$, if $\sum_{j\in\mathbb{Z}} a_j e^{-2\pi i(\beta j + \alpha\binom{j}{2})} \neq 0$ and c is such that $\Phi > 0$.
3. We assume only $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, no *Diophantine Condition* on α. Mixing is not quantitative.

Open Questions:

- Do Thm. /Cor. hold within the class of all smooth time-changes?
- Under a Diophantine conditions on the frequency, is the correlation decay polynomial in time for sufficiently smooth functions ?

Elliptic Case

Compare with:

special flows over rotations on $\mathbb{T}^n$	$\leftrightarrow$	time-changes of linear flows on $\mathbb{T}^{n+1}$
$(x_1, \dots, x_n) \xrightarrow{R_\alpha} (x_1 + \alpha_1, \dots, x_n + \alpha_n)$		$\frac{\mathrm{d}}{\mathrm{d}t}(x_1, \dots, x_{n+1}) = (\alpha_1, \dots, \alpha_{n+1})$

- $n = 1$ special flows over R_α under a smooth roof Φ are never mixing (Katok);
- $n \geq 2$ If $\underline{\alpha}$ satisfies *Diophantine Conditions*, special flows over $R_{\underline{\alpha}}$ under a *smooth* roof Φ are not mixing (KAM);
- Fayad: *There exist* rotation numbers (α_1, α_2) (very Liouville!) and an *analytic* roof function Φ such that the special flow over the rotation $(x_1, x_2) \mapsto (x_1 + \alpha_1, x_2 + \alpha_2)$ under Φ is *mixing*;

Remark: Fayad phenomenon is measure zero. In the parabolic setting smooth mixing reparametrizations exist for *all* irrational α. It's related to the existence of non trivial time-changes and obstructions to solving the cohomological equation.

Elliptic Case

Compare with:

special flows over rotations on $\mathbb{T}^n$	$\leftrightarrow$	time-changes of linear flows on $\mathbb{T}^{n+1}$
$(x_1, \dots, x_n) \xrightarrow{R_\alpha} (x_1 + \alpha_1, \dots, x_n + \alpha_n)$		$\frac{d}{dt}(x_1, \dots, x_{n+1}) = (\alpha_1, \dots, \alpha_{n+1})$

- $n = 1$ special flows over R_α under a smooth roof Φ are never mixing (Katok);
- $n \geq 2$ If $\underline{\alpha}$ satisfies *Diophantine Conditions*, special flows over $R_{\underline{\alpha}}$ under a *smooth* roof Φ are not mixing (KAM);
- Fayad: *There exist* rotation numbers (α_1, α_2) (very Liouville!) and an *analytic* roof function Φ such that the special flow over the rotation $(x_1, x_2) \mapsto (x_1 + \alpha_1, x_2 + \alpha_2)$ under Φ is *mixing*;

Remark: Fayad phenomenon is measure zero. In the parabolic setting smooth mixing reparametrizations exist for *all* irrational α. It's related to the existence of non trivial time-changes and obstructions to solving the cohomological equation.

Elliptic Case

Compare with:

special flows over rotations on $\mathbb{T}^n$	$\leftrightarrow$	time-changes of linear flows on $\mathbb{T}^{n+1}$
$(x_1, \ldots, x_n) \xrightarrow{R_\alpha} (x_1 + \alpha_1, \ldots, x_n + \alpha_n)$		$\frac{\mathrm{d}}{\mathrm{d}t}(x_1, \ldots, x_{n+1}) = (\alpha_1, \ldots, \alpha_{n+1})$

- $n = 1$ special flows over R_α under a smooth roof Φ are never mixing (Katok);
- $n \geq 2$ If $\underline{\alpha}$ satisfies *Diophantine Conditions*, special flows over $R_{\underline{\alpha}}$ under a *smooth* roof Φ are not mixing (KAM);
- Fayad: *There exist* rotation numbers (α_1, α_2) (very Liouville!) and an *analytic* roof function Φ such that the special flow over the rotation $(x_1, x_2) \mapsto (x_1 + \alpha_1, x_2 + \alpha_2)$ under Φ is *mixing*;

Remark: Fayad phenomenon is measure zero. In the parabolic setting smooth mixing reparametrizations exist for *all* irrational α. It's related to the existence of non trivial time-changes and obstructions to solving the cohomological equation.

Elliptic Case

Compare with:

special flows over rotations on $\mathbb{T}^n$	$\leftrightarrow$	time-changes of linear flows on $\mathbb{T}^{n+1}$
$(x_1, \dots, x_n) \xrightarrow{R_\alpha} (x_1 + \alpha_1, \dots, x_n + \alpha_n)$		$\frac{\mathrm{d}}{\mathrm{d}t}(x_1, \dots, x_{n+1}) = (\alpha_1, \dots, \alpha_{n+1})$

- $n = 1$ special flows over R_α under a smooth roof Φ are never mixing (Katok);
- $n \geq 2$ If $\underline{\alpha}$ satisfies *Diophantine Conditions*, special flows over $R_{\underline{\alpha}}$ under a *smooth* roof Φ are not mixing (KAM);
- Fayad: *There exist* rotation numbers (α_1, α_2) (very Liouville!) and an *analytic* roof function Φ such that the special flow over the rotation $(x_1, x_2) \mapsto (x_1 + \alpha_1, x_2 + \alpha_2)$ under Φ is *mixing*;

Remark: Fayad phenomenon is measure zero. In the parabolic setting smooth mixing reparametrizations exist for *all* irrational α. It's related to the existence of non trivial time-changes and obstructions to solving the cohomological equation.

Elliptic Case

Compare with:

special flows over rotations on $\mathbb{T}^n$	$\leftrightarrow$	time-changes of linear flows on $\mathbb{T}^{n+1}$
$(x_1, \dots, x_n) \xrightarrow{R_\alpha} (x_1 + \alpha_1, \dots, x_n + \alpha_n)$		$\frac{\mathrm{d}}{\mathrm{d}t}(x_1, \dots, x_{n+1}) = (\alpha_1, \dots, \alpha_{n+1})$

- $n = 1$ special flows over R_α under a smooth roof Φ are never mixing (Katok);
- $n \geq 2$ If $\underline{\alpha}$ satisfies *Diophantine Conditions*, special flows over $R_{\underline{\alpha}}$ under a *smooth* roof Φ are not mixing (KAM);
- Fayad: *There exist* rotation numbers (α_1, α_2) (very Liouville!) and an *analytic* roof function Φ such that the special flow over the rotation $(x_1, x_2) \mapsto (x_1 + \alpha_1, x_2 + \alpha_2)$ under Φ is *mixing*;

Remark: Fayad phenomenon is measure zero. In the parabolic setting smooth mixing reparametrizations exist for *all* irrational α. It's related to the existence of non trivial time-changes and obstructions to solving the cohomological equation.

Elliptic Case

Compare with:

special flows over rotations on $\mathbb{T}^n$	$\leftrightarrow$	time-changes of linear flows on $\mathbb{T}^{n+1}$
$(x_1, \dots, x_n) \xrightarrow{R_\alpha} (x_1 + \alpha_1, \dots, x_n + \alpha_n)$		$\frac{\mathrm{d}}{\mathrm{d}t}(x_1, \dots, x_{n+1}) = (\alpha_1, \dots, \alpha_{n+1})$

- $n = 1$ special flows over R_α under a smooth roof Φ are never mixing (Katok);
- $n \geq 2$ If $\underline{\alpha}$ satisfies *Diophantine Conditions*, special flows over $R_{\underline{\alpha}}$ under a *smooth* roof Φ are not mixing (KAM);
- Fayad: *There exist* rotation numbers (α_1, α_2) (very Liouville!) and an *analytic* roof function Φ such that the special flow over the rotation $(x_1, x_2) \mapsto (x_1 + \alpha_1, x_2 + \alpha_2)$ under Φ is *mixing*;

Remark: Fayad phenomenon is measure zero. In the parabolic setting smooth mixing reparametrizations exist for *all* irrational α. It's related to the existence of non trivial time-changes and obstructions to solving the cohomological equation.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^\perp(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x,\cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^\perp$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^\infty(\mathbb{T}^2)$ is a dense subspace (e. g. for $\|\cdot\|_\infty$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^{\perp}(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x, \cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^{\perp}$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ is a dense subspace (e. g. for $\|\cdot\|_{\infty}$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^{\perp}(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x,\cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^{\perp}$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ is a dense subspace (e. g. for $\|\cdot\|_{\infty}$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^{\perp}(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x,\cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^{\perp}$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ is a dense subspace (e. g. for $\|\cdot\|_{\infty}$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^{\perp}(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x,\cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^{\perp}$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ is a dense subspace (e. g. for $\|\cdot\|_{\infty}$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^{\perp}(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x, \cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^{\perp}$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ is a dense subspace (e. g. for $\| \cdot \|_{\infty}$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^{\perp}(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x,\cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^{\perp}$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ is a dense subspace (e. g. for $\|\cdot\|_{\infty}$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Class of Mixing Roof Functions

Let $\Phi \in L^2(\mathbb{T}^2)$. Introduce the following notation:

$$\phi(x,y) := \Phi(x,y) - \int \Phi(x,y)dy \qquad \phi^{\perp}(x) := \int \Phi(x,y)dy$$

The class $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ contains all *trigonometric polynomials* in x, y.

Definition (Roofs class $\mathcal{R}$)

The function $\Phi \in \mathcal{R}$ iff Φ is continuous, for each $x \in \mathbb{T}$, $\Phi(x, \cdot)$ is a trigonometric polynomial of degree at most d on $\mathbb{T}$ and $\Phi \in \mathcal{P}$ and $\phi^{\perp}$ is a trigonometric polynomial on $\mathbb{T}$.

Remark: $\mathcal{R} \subset C^{\infty}(\mathbb{T}^2)$ is a dense subspace (e. g. for $\|\cdot\|_{\infty}$).

$\Phi : \Sigma \to \mathbb{R}$ is called a measurable (smooth) *coboundary* for $f : \Sigma \to \Sigma$ iff $\exists$ measurable (smooth) function $u : \Sigma \to \mathbb{R}$, called the *transfer function*, s. t. $\Phi = u \circ f - u$.

Definition (Trivial roofs $\mathcal{T}_f$ and mixing roofs $\mathcal{M}_f$)

A function Φ belongs to $\mathcal{T}_f$ iff $\Phi \in \mathcal{R}$ and its projection ϕ is a measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$. Set $\mathcal{M}_f := \mathcal{R} \setminus \mathcal{T}_f$, so that Φ belongs to $\mathcal{M}_f$ iff $\Phi \in \mathcal{R}$ and ϕ is *not* a measurable coboundary.

Cocycle Effectiveness

The condition $\Phi \in \mathcal{M}_f$ iff ϕ is a *not* measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is virtually impossible to check explicitely.

The class $\mathcal{M}_f$ is *explicit* becouse we can also prove:

Proposition

If ϕ is regular ($f \in W^s(\mathbb{T}^2)$, standard Sobolev space with $s > 3$)), then ϕ is a measurable coboundary for a skew-shift f on $\mathbb{T}^2$ with a measurable transfer function if and only if ϕ is a smooth coboundary for f.

One can explicitely check if f is a smooth coboundary.

Lemma

There exists countably many (explicit) invariant distributions $D_{(m,n)}$ such that ϕ is a smooth couboundary iff $D_{(m,n)}(\phi) = 0$ for all m, n.

Invariant distributions $D_{(m,n)}$, where $m \in \mathbb{Z}\setminus\{0\}$, $n \in \mathbb{Z}_{|n|}$:

$$D_{(m,n)}(e_{a,b}) := \begin{cases} e^{-2\pi i[(\alpha m+\beta n)j+\alpha n\binom{j}{2}]} & \text{if } (a,b) = (m+jn, n); \\ 0 & \text{otherwise.} \end{cases}$$

where $e_{a,b}(x,y) := \exp[2\pi i(ax+by)]$.

Cocycle Effectiveness

The condition $\Phi \in \mathcal{M}_f$ iff ϕ is a *not* measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is virtually impossible to check explicitely.

The class $\mathcal{M}_f$ is *explicit* becouse we can also prove:

Proposition

If ϕ is regular ($f \in W^s(\mathbb{T}^2)$, standard Sobolev space with $s > 3$)), then ϕ is a measurable coboundary for a skew-shift f on $\mathbb{T}^2$ with a measurable transfer function if and only if ϕ is a smooth coboundary for f.

One can explicitely check if f is a smooth coboundary.

Lemma

There exists countably many (explicit) invariant distributions $D_{(m,n)}$ such that ϕ is a smooth couboundary iff $D_{(m,n)}(\phi) = 0$ for all m, n.

Invariant distributions $D_{(m,n)}$, where $m \in \mathbb{Z}\setminus\{0\}, n \in \mathbb{Z}_{|n|}$:

$$D_{(m,n)}(e_{a,b}) := \begin{cases} e^{-2\pi i[(\alpha m+\beta n)j+\alpha n\binom{j}{2}]} & \text{if } (a,b) = (m+jn, n)\,; \\ 0 & \text{otherwise.} \end{cases}$$

where $e_{a,b}(x, y) := \exp[2\pi i(ax + by)]$.

Cocycle Effectiveness

The condition $\Phi \in \mathcal{M}_f$ iff ϕ is a *not* measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is virtually impossible to check explicitely.

The class $\mathcal{M}_f$ is *explicit* becouse we can also prove:

Proposition

If ϕ is regular ($f \in W^s(\mathbb{T}^2)$, standard Sobolev space with $s > 3$)), then ϕ is a measurable coboundary for a skew-shift f on $\mathbb{T}^2$ with a measurable transfer function if and only if ϕ is a smooth coboundary for f.

One can explicitely check if f is a smooth coboundary.

Lemma

There exists countably many (explicit) invariant distributions $D_{(m,n)}$ such that ϕ is a smooth couboundary iff $D_{(m,n)}(\phi) = 0$ for all m, n.

Invariant distributions $D_{(m,n)}$, where $m \in \mathbb{Z}\backslash\{0\}, n \in \mathbb{Z}_{|n|}$:

$$D_{(m,n)}(e_{a,b}) := \begin{cases} e^{-2\pi i[(\alpha m+\beta n)j+\alpha n\binom{j}{2}]} & \text{if } (a,b) = (m+jn, n); \\ 0 & \text{otherwise.} \end{cases}$$

where $e_{a,b}(x,y) := \exp[2\pi i(ax+by)]$.

Cocycle Effectiveness

The condition $\Phi \in \mathcal{M}_f$ iff ϕ is a *not* measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is virtually impossible to check explicitely.

The class $\mathcal{M}_f$ is *explicit* becouse we can also prove:

Proposition

If ϕ is regular ($f \in W^s(\mathbb{T}^2)$, standard Sobolev space with $s > 3$)), then ϕ is a measurable coboundary for a skew-shift f on $\mathbb{T}^2$ with a measurable transfer function if and only if ϕ is a smooth coboundary for f.

One can explicitely check if f is a smooth coboundary.

Lemma

There exists countably many (explicit) invariant distributions $D_{(m,n)}$ such that ϕ is a smooth couboundary iff $D_{(m,n)}(\phi) = 0$ for all m, n.

Invariant distributions $D_{(m,n)}$, where $m \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{Z}_{|n|}$:

$$D_{(m,n)}(e_{a,b}) := \begin{cases} e^{-2\pi i[(\alpha m + \beta n)j + \alpha n \binom{j}{2}]} & \text{if } (a,b) = (m + jn, n); \\ 0 & \text{otherwise.} \end{cases}$$

where $e_{a,b}(x, y) := \exp[2\pi i(ax + by)]$.

Cocycle Effectiveness

The condition $\Phi \in \mathcal{M}_f$ iff ϕ is a *not* measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is virtually impossible to check explicitely.

The class $\mathcal{M}_f$ is *explicit* becouse we can also prove:

Proposition

If ϕ is regular ($f \in W^s(\mathbb{T}^2)$, standard Sobolev space with $s > 3$)), then ϕ is a measurable coboundary for a skew-shift f on $\mathbb{T}^2$ with a measurable transfer function if and only if ϕ is a smooth coboundary for f.

One can explicitely check if f is a smooth coboundary.

Lemma

There exists countably many (explicit) invariant distributions $D_{(m,n)}$ such that ϕ is a smooth couboundary iff $D_{(m,n)}(\phi) = 0$ for all m, n.

Invariant distributions $D_{(m,n)}$, where $m \in \mathbb{Z} \backslash \{0\}, n \in \mathbb{Z}_{|n|}$:

$$D_{(m,n)}(e_{a,b}) := \begin{cases} e^{-2\pi i[(\alpha m+\beta n)j+\alpha n\binom{j}{2}]} & \text{if } (a,b) = (m+jn, n)\,; \\ 0 & \text{otherwise.} \end{cases}$$

where $e_{a,b}(x,y) := \exp[2\pi i(ax+by)]$.

Cocycle Effectiveness

The condition $\Phi \in \mathcal{M}_f$ iff ϕ is a *not* measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is virtually impossible to check explicitely.

The class $\mathcal{M}_f$ is *explicit* becouse we can also prove:

Proposition

If ϕ is regular ($f \in W^s(\mathbb{T}^2)$, standard Sobolev space with $s > 3$)), then ϕ is a measurable coboundary for a skew-shift f on $\mathbb{T}^2$ with a measurable transfer function if and only if ϕ is a smooth coboundary for f.

One can explicitely check if f is a smooth coboundary.

Lemma

There exists countably many (explicit) invariant distributions $D_{(m,n)}$ such that ϕ is a smooth couboundary iff $D_{(m,n)}(\phi) = 0$ for all m, n.

- The Lemma is known since work by Katok in the '80s on the cohomological equation for skew-shifts.
- The proof of the Proposition is based on the quantitative estimates on equidistribution of the Heisenberg nilflows by Flaminio and Forni.

Cocycle Effectiveness

The condition $\Phi \in \mathcal{M}_f$ iff ϕ is a *not* measurable coboundary for the map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is virtually impossible to check explicitely.

The class $\mathcal{M}_f$ is *explicit* becouse we can also prove:

Proposition

If ϕ is regular ($f \in W^s(\mathbb{T}^2)$, standard Sobolev space with $s > 3$)), then ϕ is a measurable coboundary for a skew-shift f on $\mathbb{T}^2$ with a measurable transfer function if and only if ϕ is a smooth coboundary for f.

One can explicitely check if f is a smooth coboundary.

Lemma

There exists countably many (explicit) invariant distributions $D_{(m,n)}$ such that ϕ is a smooth couboundary iff $D_{(m,n)}(\phi) = 0$ for all m, n.

- The Lemma is known since work by Katok in the '80s on the cohomological equation for skew-shifts.
- The proof of the Proposition is based on the quantitative estimates on equidistribution of the Heisenberg nilflows by Flaminio and Forni.

Remarks on higer dimensions

Open Questions:

- Does the main Theorem extend to more general nilflows?
- Does the main Theorem extend to special flows over linear skew-shift on $\mathbb{T}^n$ with $n > 2$? (they correspond to a class of nilflows known as filiphorm nilflows)

The main Theorem splits as we saw in these two parts:

1. *Mixing class:* there exists a class $\mathcal{M}_f$ (defined in terms of ϕ not a *measurable coboundary*) such that $\Phi \in \mathcal{M}_f$ implies mixing;
2. *Cocycle Effectiveness* the class $\mathcal{M}_f$ can be described explicitely since ϕ is a *measurable coboundary*) iff it is a smooth coboundary.

- We believe that Part 1 does generalize to linear skew-shift on $\mathbb{T}^n$ with $n > 2$.
- Part 2 relies on estimates currently known only for $n = 2$. (Flaminio-Forni sharp estimates for Heisenberg nilflows, related to bounds on Weyl sums for quadratic polynomials by Marklof, Fiedler-Jurkat)

Remarks on higer dimensions

Open Questions:

- Does the main Theorem extend to more general nilflows?
- Does the main Theorem extend to special flows over linear skew-shift on $\mathbb{T}^n$ with $n > 2$? (they correspond to a class of nilflows known as filiphorm nilflows)

The main Theorem splits as we saw in these two parts:

1. *Mixing class:* there exists a class $\mathcal{M}_f$ (defined in terms of ϕ not a *measurable coboundary*) such that $\Phi \in \mathcal{M}_f$ implies mixing;
2. *Cocycle Effectiveness* the class $\mathcal{M}_f$ can be described explicitely since ϕ is a *measurable coboundary*) iff it is a smooth coboundary.

- We believe that Part 1 does generalize to linear skew-shift on $\mathbb{T}^n$ with $n > 2$.
- Part 2 relies on estimates currently known only for $n = 2$. (Flaminio-Forni sharp estimates for Heisenberg nilflows, related to bounds on Weyl sums for quadratic polynomials by Marklof, Fiedler-Jurkat)

Remarks on higer dimensions

Open Questions:

- Does the main Theorem extend to more general nilflows?
- Does the main Theorem extend to special flows over linear skew-shift on $\mathbb{T}^n$ with $n > 2$? (they correspond to a class of nilflows known as filiphorm nilflows)

The main Theorem splits as we saw in these two parts:

1. *Mixing class:* there exists a class $\mathcal{M}_f$ (defined in terms of ϕ not a *measurable coboundary*) such that $\Phi \in \mathcal{M}_f$ implies mixing;
2. *Cocycle Effectiveness* the class $\mathcal{M}_f$ can be described explicitely since ϕ is a *measurable coboundary*) iff it is a smooth coboundary.

- We believe that Part 1 does generalize to linear skew-shift on $\mathbb{T}^n$ with $n > 2$.
- Part 2 relies on estimates currently known only for $n = 2$. (Flaminio-Forni sharp estimates for Heisenberg nilflows, related to bounds on Weyl sums for quadratic polynomials by Marklof, Fiedler-Jurkat)

Remarks on higer dimensions

Open Questions:

- Does the main Theorem extend to more general nilflows?
- Does the main Theorem extend to special flows over linear skew-shift on $\mathbb{T}^n$ with $n > 2$? (they correspond to a class of nilflows known as filiphorm nilflows)

The main Theorem splits as we saw in these two parts:

1. *Mixing class:* there exists a class $\mathcal{M}_f$ (defined in terms of ϕ not a *measurable coboundary*) such that $\Phi \in \mathcal{M}_f$ implies mixing;
2. *Cocycle Effectiveness* the class $\mathcal{M}_f$ can be described explicitely since ϕ is a *measurable coboundary*) iff it is a smooth coboundary.

- We believe that Part 1 does generalize to linear skew-shift on $\mathbb{T}^n$ with $n > 2$.
- Part 2 relies on estimates currently known only for $n = 2$. (Flaminio-Forni sharp estimates for Heisenberg nilflows, related to bounds on Weyl sums for quadratic polynomials by Marklof, Fiedler-Jurkat)

Remarks on higer dimensions

Open Questions:

- Does the main Theorem extend to more general nilflows?
- Does the main Theorem extend to special flows over linear skew-shift on $\mathbb{T}^n$ with $n > 2$? (they correspond to a class of nilflows known as filiphorm nilflows)

The main Theorem splits as we saw in these two parts:

1. *Mixing class:* there exists a class $\mathcal{M}_f$ (defined in terms of ϕ not a *measurable coboundary*) such that $\Phi \in \mathcal{M}_f$ implies mixing;
2. *Cocycle Effectiveness* the class $\mathcal{M}_f$ can be described explicitely since ϕ is a *measurable coboundary*) iff it is a smooth coboundary.

- We believe that Part 1 does generalize to linear skew-shift on $\mathbb{T}^n$ with $n > 2$.
- Part 2 relies on estimates currently known only for $n = 2$. (Flaminio-Forni sharp estimates for Heisenberg nilflows, related to bounds on Weyl sums for quadratic polynomials by Marklof, Fiedler-Jurkat)

Sketch $\Phi \in \mathcal{M}_f \Rightarrow$ mixing

Assume that $\Phi \in \mathcal{M}_f$, thus $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$ is not a measurable coboundary.

Let $\phi_n = \sum_{i=0}^{n-1} \phi \circ f^n$ denote Birkhoff sums of the function ϕ along the skew shift f.
The crucial ingredient in the proof of mixing is given by the a result on the growth of Birkhoff sums of the skew-shift.

The proof splits in two steps.

- Step 1: Stretch of Birkhoff sums
  ϕ not couboundary $\Rightarrow$ for each $C > 1$,

$$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| < C) \xrightarrow{n\to\infty} 0\,.$$

- Step 2: Stretch $\Rightarrow$ Mixing
  through a geometric mixing mechanism (next slides).

Remark: the mixing mechanism is similar to the one used by Fayad in the elliptic Liouvillean case and in the proof of mixing in multi-valued Hamiltonian flows on surfaces with saddle loops (U'07).

Sketch $\Phi \in \mathcal{M}_f \Rightarrow$ mixing

Assume that $\Phi \in \mathcal{M}_f$, thus $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$ is not a measurable coboundary.

Let $\phi_n = \sum_{i=0}^{n-1} \phi \circ f^n$ denote Birkhoff sums of the function ϕ along the skew shift f.
The crucial ingredient in the proof of mixing is given by the a result on the growth of Birkhoff sums of the skew-shift.

The proof splits in two steps.

- Step 1: Stretch of Birkhoff sums
  ϕ not couboundary $\Rightarrow$ for each $C > 1$,

$$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| < C) \xrightarrow{n\to\infty} 0\,.$$

- Step 2: Stretch $\Rightarrow$ Mixing
  through a geometric mixing mechanism (next slides).

Remark: the mixing mechanism is similar to the one used by Fayad in the elliptic Liouvillean case and in the proof of mixing in multi-valued Hamiltonian flows on surfaces with saddle loops (U'07)

Sketch $\Phi \in \mathcal{M}_f \Rightarrow$ mixing

Assume that $\Phi \in \mathcal{M}_f$, thus $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$ is not a measurable coboundary.

Let $\phi_n = \sum_{i=0}^{n-1} \phi \circ f^n$ denote Birkhoff sums of the function ϕ along the skew shift f.
The crucial ingredient in the proof of mixing is given by the a result on the growth of Birkhoff sums of the skew-shift.

The proof splits in two steps.

- Step 1: Stretch of Birkhoff sums
  ϕ not couboundary $\Rightarrow$ for each $C > 1$,

$$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| < C) \xrightarrow{n\to\infty} 0\,.$$

- Step 2: Stretch $\Rightarrow$ Mixing
  through a geometric mixing mechanism (next slides).

Remark: the mixing mechanism is similar to the one used by Fayad in the elliptic Liouvillean case and in the proof of mixing in multi-valued Hamiltonian flows on surfaces with saddle loops (U'07)

Sketch $\Phi \in \mathcal{M}_f \Rightarrow$ mixing

Assume that $\Phi \in \mathcal{M}_f$, thus $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$ is not a measurable coboundary.

Let $\phi_n = \sum_{i=0}^{n-1} \phi \circ f^n$ denote Birkhoff sums of the function ϕ along the skew shift f.
The crucial ingredient in the proof of mixing is given by the a result on the growth of Birkhoff sums of the skew-shift.

The proof splits in two steps.

- Step 1: Stretch of Birkhoff sums
  ϕ not couboundary $\Rightarrow$ for each $C > 1$,

$$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| < C) \xrightarrow{n\to\infty} 0\,.$$

- Step 2: Stretch $\Rightarrow$ Mixing
  through a geometric mixing mechanism (next slides).

Remark: the mixing mechanism is similar to the one used by Fayad in the elliptic Liouvillean case and in the proof of mixing in multi-valued Hamiltonian flows on surfaces with saddle loops (U'07)

Sketch $\Phi \in \mathcal{M}_f \Rightarrow$ mixing

Assume that $\Phi \in \mathcal{M}_f$, thus $\phi(x,y) = \Phi(x,y) - \int \Phi(x,y)\,\mathrm{d}y$ is not a measurable coboundary.

Let $\phi_n = \sum_{i=0}^{n-1} \phi \circ f^n$ denote Birkhoff sums of the function ϕ along the skew shift f.
The crucial ingredient in the proof of mixing is given by the a result on the growth of Birkhoff sums of the skew-shift.

The proof splits in two steps.

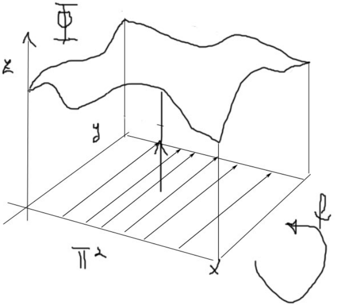
- Step 1: Stretch of Birkhoff sums
  ϕ not couboundary $\Rightarrow$ for each $C > 1$,

$$\mathrm{Leb}((x,y)\, s.t. \quad |\phi_n(x,y)| < C) \xrightarrow{n\to\infty} 0\,.$$

- Step 2: Stretch $\Rightarrow$ Mixing
  through a geometric mixing mechanism (next slides).

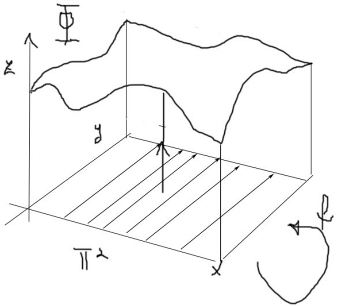
Remark: the mixing mechanism is similar to the one used by Fayad in the elliptic Liouvillean case and in the proof of mixing in multi-valued Hamiltonian flows on surfaces with saddle loops (U'07)

Mixing mechanism picture



Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$. For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

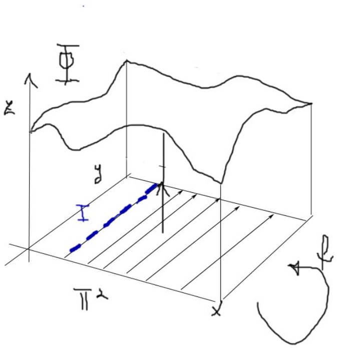
Mixing mechanism picture



Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$.

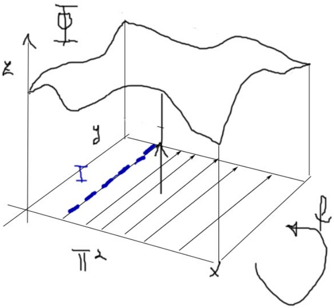
For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

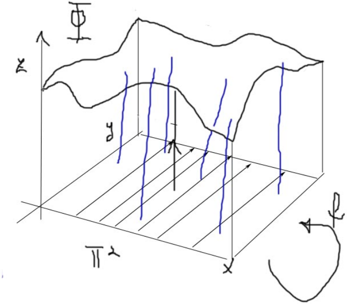
Mixing mechanism picture



Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$.
For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

Mixing mechanism picture





Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$. For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

image $f_t^{\Phi}(I)$ for $t >> 1$ each interval I looks as above (stretched in the z direction and shadows a long orbit of f)

Step 1: Stretch of Birkhoff sums

ϕ not a coboundary $\Rightarrow \forall C > 1$, $\lim_{n\to\infty} \mathrm{Leb}(|\phi_n| < C) = 0$.

Sketch:

1. Since f is uniquely ergodic, by a standard Gottschalk-Hedlund technique, $\forall C > 1, \quad \forall (x, y) \in \mathbb{T}^2$,

$$\frac{1}{N}\#\{0 \leq n \leq N-1, : \phi_n(x,y)| < C\} \xrightarrow{N\to\infty} 0;$$

2. Integrating we get: $\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{Leb}(|\phi_n| < C) \xrightarrow{N\to\infty} 0$.
3. Using the explicit form of the skew-shift we get the Decoupling lemma: $\forall \epsilon' > 0$, $\exists C' > 1$, $\epsilon'' > 0$ s.t. $\forall n \geq 1$ s.t. $\mathrm{Leb}(|\phi_n| < C') < \epsilon''$, $\forall N \geq N_0(C, \epsilon', n)$, we have

$$\mathrm{Leb}(|\phi_N \circ f^n - \phi_N| < 2C) < \epsilon'$$

.

4. Combining $3 + 4$ we get the non-averaged stretch estimate.

Step 1: Stretch of Birkhoff sums

ϕ not a coboundary $\Rightarrow \forall C > 1$, $\lim_{n\to\infty} \mathrm{Leb}(|\phi_n| < C) = 0$.

Sketch:

1. Since f is uniquely ergodic, by a standard Gottschalk-Hedlund technique, $\forall C > 1, \quad \forall (x,y) \in \mathbb{T}^2$,

$$\frac{1}{N}\#\{0 \leq n \leq N-1, : \phi_n(x,y)| < C\} \xrightarrow{N\to\infty} 0;$$

2. Integrating we get: $\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{Leb}(|\phi_n| < C) \xrightarrow{N\to\infty} 0.$
3. Using the explicit form of the skew-shift we get the Decoupling lemma: $\forall \epsilon' > 0$, $\exists C' > 1$, $\epsilon'' > 0$ s.t. $\forall n \geq 1$ s.t. $\mathrm{Leb}(|\phi_n| < C') < \epsilon''$, $\forall N \geq N_0(C, \epsilon', n)$, we have

$$\mathrm{Leb}(|\phi_N \circ f^n - \phi_N| < 2C) < \epsilon'$$

.

4. Combining $3 + 4$ we get the non-averaged stretch estimate.

Step 1: Stretch of Birkhoff sums

ϕ not a coboundary $\Rightarrow \forall C > 1$, $\lim_{n\to\infty} \mathrm{Leb}(|\phi_n| < C) = 0$.

Sketch:

1. Since f is uniquely ergodic, by a standard Gottschalk-Hedlund technique, $\forall C > 1, \quad \forall (x,y) \in \mathbb{T}^2$,

$$\frac{1}{N}\#\{0 \le n \le N-1, : \phi_n(x,y)| < C\} \xrightarrow{N\to\infty} 0;$$

2. Integrating we get: $\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{Leb}(|\phi_n| < C) \xrightarrow{N\to\infty} 0$.
3. Using the explicit form of the skew-shift we get the Decoupling lemma: $\forall \epsilon' > 0$, $\exists C' > 1$, $\epsilon'' > 0$ s.t. $\forall n \ge 1$ s.t. $\mathrm{Leb}(|\phi_n| < C') < \epsilon''$, $\forall N \ge N_0(C, \epsilon', n)$, we have

$$\mathrm{Leb}(|\phi_N \circ f^n - \phi_N| < 2C) < \epsilon'$$

.

4. Combining $3+4$ we get the non-averaged stretch estimate.

Step 1: Stretch of Birkhoff sums

ϕ not a coboundary $\Rightarrow \forall C > 1$, $\lim_{n\to\infty} \mathrm{Leb}(|\phi_n| < C) = 0$.

Sketch:

1. Since f is uniquely ergodic, by a standard Gottschalk-Hedlund technique, $\forall C > 1, \quad \forall (x,y) \in \mathbb{T}^2$,

$$\frac{1}{N} \#\{0 \leq n \leq N-1, : \phi_n(x,y)| < C\} \xrightarrow{N\to\infty} 0;$$

2. Integrating we get: $\frac{1}{N} \sum_{n=0}^{N-1} \mathrm{Leb}(|\phi_n| < C) \xrightarrow{N\to\infty} 0$.
3. Using the explicit form of the skew-shift we get the
   Decoupling lemma: $\forall \epsilon' > 0$, $\exists C' > 1$, $\epsilon'' > 0$ s.t. $\forall n \geq 1$ s.t. $\mathrm{Leb}(|\phi_n| < C') < \epsilon''$, $\forall N \geq N_0(C, \epsilon', n)$, we have

$$\mathrm{Leb}(|\phi_N \circ f^n - \phi_N| < 2C) < \epsilon'$$

.

4. Combining $3 + 4$ we get the non-averaged stretch estimate.

Step 1: Stretch of Birkhoff sums

ϕ not a coboundary $\Rightarrow \forall C > 1$, $\lim_{n\to\infty} \mathrm{Leb}(|\phi_n| < C) = 0$.

Sketch:

1. Since f is uniquely ergodic, by a standard Gottschalk-Hedlund technique, $\forall C > 1, \quad \forall (x,y) \in \mathbb{T}^2$,

$$\frac{1}{N}\#\{0 \leq n \leq N-1, : \phi_n(x,y)| < C\} \xrightarrow{N\to\infty} 0;$$

2. Integrating we get: $\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{Leb}(|\phi_n| < C) \xrightarrow{N\to\infty} 0$.
3. Using the explicit form of the skew-shift we get the

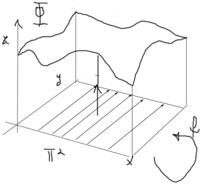
   Decoupling lemma: $\forall \epsilon' > 0$, $\exists C' > 1$, $\epsilon'' > 0$ s.t. $\forall n \geq 1$ s.t. $\mathrm{Leb}(|\phi_n| < C') < \epsilon''$, $\forall N \geq N_0(C, \epsilon', n)$, we have

$$\mathrm{Leb}(|\phi_N \circ f^n - \phi_N| < 2C) < \epsilon'$$

.

4. Combining $3 + 4$ we get the non-averaged stretch estimate.

Step 2: from stretch to mixing



The special flow $f^\Phi = \{f_t^\Phi\}_{t\in\mathbb{R}}$ acts by:

$$f_t^\Phi((x,y),0) = \left(f^{n_t(x,y)}, t - \Phi_{n_t(x,y)}\right).$$

where $n_t(x,y) := \max\{n \in \mathbb{N} : \Phi_n(x,y) < t\}$.

Remark 1: f is an isometry in the y-direction: $y \mapsto f(\cdot, x+y)$;

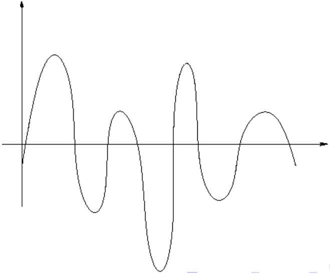
Remark 2: we have $\frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y}$ (since $\phi = \Phi - \int \Phi \, \mathrm{d}y$).

Thus $\mathrm{Leb}(|\phi_n| < C) = \mathrm{Leb}(|\Phi_n| < C)$.

Since ϕ is a trigonometric polynomial,

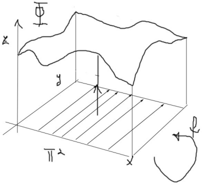
$|\phi|$ large $\Rightarrow |\frac{\partial \phi}{\partial y}|$ large

Throw away intervals where it is small to construct good I.



Step 2: from stretch to mixing

The special flow $f^{\Phi} = \{f_t^{\Phi}\}_{t\in\mathbb{R}}$ acts by:



$$f_t^{\Phi}((x,y),0) = \left(f^{n_t(x,y)}, t - \Phi_{n_t(x,y)}\right).$$

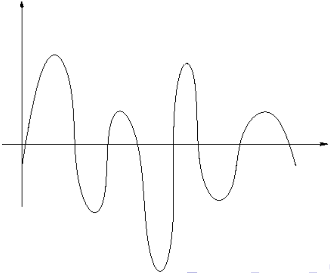
where $n_t(x,y) := \max\{n \in \mathbb{N} :\ \Phi_n(x,y) < t\}$.

Remark 1: f is an isometry in the y-direction: $y \mapsto f(\cdot, x+y)$;

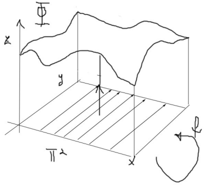
Remark 2: we have $\frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y}$ (since $\phi = \Phi - \int \Phi \, \mathrm{d}y$).

Thus $\mathrm{Leb}(|\phi_n| < C) = \mathrm{Leb}(|\Phi_n| < C)$.

Since ϕ is a trigonometric polynomial,
$|\phi|$ large $\Rightarrow |\frac{\partial \phi}{\partial y}|$ large
Throw away intervals where it is small to construct good I.



Step 2: from stretch to mixing



The special flow $f^{\Phi} = \{f_t^{\Phi}\}_{t\in\mathbb{R}}$ acts by:

$$f_t^{\Phi}((x,y),0) = \left(f^{n_t(x,y)}, t - \Phi_{n_t(x,y)}\right).$$

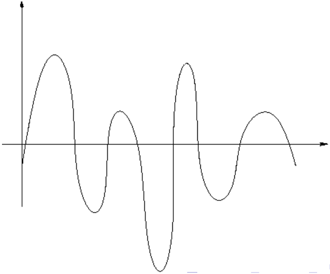
where $n_t(x,y) := \max\{n \in \mathbb{N} : \Phi_n(x,y) < t\}$.

Remark 1: f is an isometry in the y-direction: $y \mapsto f(\cdot, x+y)$;

Remark 2: we have $\frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y}$ (since $\phi = \Phi - \int \Phi \, \mathrm{d}y$).

Thus $\mathrm{Leb}(|\phi_n| < C) = \mathrm{Leb}(|\Phi_n| < C)$.

Since ϕ is a trigonometric polynomial,
$|\phi|$ large $\Rightarrow |\frac{\partial \phi}{\partial y}|$ large
Throw away intervals where it is small to construct good I.

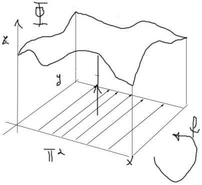


Step 2: from stretch to mixing

The special flow $f^\Phi = \{f_t^\Phi\}_{t\in\mathbb{R}}$ acts by:

$$f_t^\Phi((x,y),0) = \left(f^{n_t(x,y)}, t - \Phi_{n_t(x,y)}\right).$$

where $n_t(x,y) := \max\{n \in \mathbb{N} : \Phi_n(x,y) < t\}$.

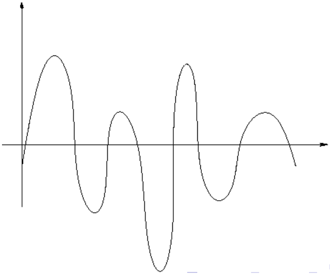


Remark 1: f is an isometry in the y-direction: $y \mapsto f(\cdot, x+y)$;

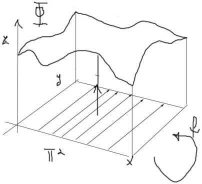
Remark 2: we have $\frac{\partial\Phi}{\partial y} = \frac{\partial\phi}{\partial y}$ (since $\phi = \Phi - \int \Phi \, \mathrm{d}y$).

Thus $\mathrm{Leb}(|\phi_n| < C) = \mathrm{Leb}(|\Phi_n| < C)$.

Since ϕ is a trigonometric polynomial,
$|\phi|$ large $\Rightarrow$ $|\frac{\partial\phi}{\partial y}|$ large
Throw away intervals where it is small to construct good I.



Step 2: from stretch to mixing



The special flow $f^{\Phi} = \{f_t^{\Phi}\}_{t\in\mathbb{R}}$ acts by:

$$f_t^{\Phi}((x,y),0) = \left(f^{n_t(x,y)}, t - \Phi_{n_t(x,y)}\right).$$

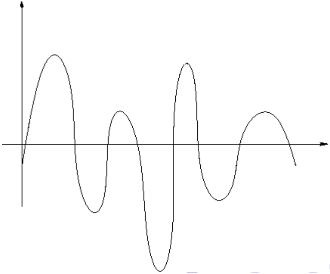
where $n_t(x,y) := \max\{n \in \mathbb{N} : \Phi_n(x,y) < t\}$.

Remark 1: f is an isometry in the y-direction: $y \mapsto f(\cdot, x+y)$;

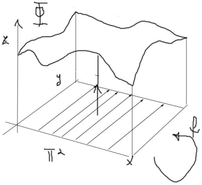
Remark 2: we have $\frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y}$ (since $\phi = \Phi - \int \Phi \, \mathrm{d}y$).

Thus $\mathrm{Leb}(|\phi_n| < C) = \mathrm{Leb}(|\Phi_n| < C)$.

Since ϕ is a trigonometric polynomial,
$|\phi|$ large $\Rightarrow |\frac{\partial \phi}{\partial y}|$ large
Throw away intervals where it is small to construct good I.



Step 2: from stretch to mixing



The special flow $f^\Phi = \{f_t^\Phi\}_{t\in\mathbb{R}}$ acts by:

$$f_t^\Phi((x,y),0) = \left(f^{n_t(x,y)}, t - \Phi_{n_t(x,y)}\right).$$

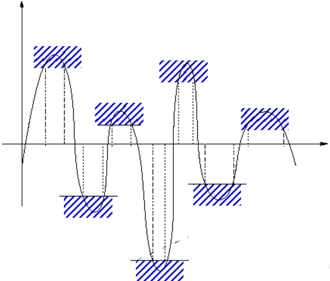
where $n_t(x,y) := \max\{n \in \mathbb{N} : \Phi_n(x,y) < t\}$.

Remark 1: f is an isometry in the y-direction: $y \mapsto f(\cdot, x+y)$;

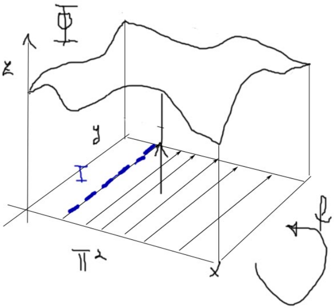
Remark 2: we have $\frac{\partial \Phi}{\partial y} = \frac{\partial \phi}{\partial y}$ (since $\phi = \Phi - \int \Phi \, \mathrm{d}y$).

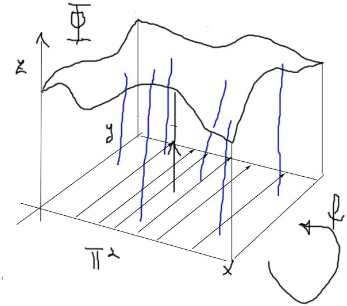
Thus $\mathrm{Leb}(|\phi_n| < C) = \mathrm{Leb}(|\Phi_n| < C)$.

Since ϕ is a trigonometric polynomial,
$|\phi|$ large $\Rightarrow |\frac{\partial \phi}{\partial y}|$ large
Throw away intervals where it is small to construct good I.



Mixing mechanism picture





Consider y-fibers $[0,1] \times \{y\} \subset \mathbb{T}^2$. For each $t > 0$ Cover large set of each fiber for large set of y with intervals I s.t.

image $f_t^{\Phi}(I)$ for $t >> 1$ each interval I looks as above (stretched in the z direction and shadows a long orbit of f)

Flaminio-Forni estimates

Theorem (Upper bounds)

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be any irrational number and let $s > 3$. There exist a constant $M_s > 0$ and a (positively) diverging sequence $\{N_\ell\}_{\ell \in \mathbb{N}}$ (depending on α) such that, for all $\Phi \in W^s(\mathbb{T}^2)$ with $\phi^\perp = 0$ and for all $(x, y) \in \mathbb{T}^2$,

$$\frac{1}{N_\ell^{1/2}} \left| \sum_{k=0}^{N_\ell - 1} \Phi \circ f^k(x, y) \right| \leq M_s \|\Phi\|_s \,. \tag{2}$$

(The theorem follows from Flaminio-Forni)

Conversely, from the explicit solutions of the cohomological equation, one can get:

Lemma (Lower bounds)

If Φ is not a smooth couboundary, there exists a constant $C_s(f) > 0$ such that

$$C_s(f)^{-1} |D_{(m,n)}(\Phi)| \leq \liminf_{N \to +\infty} \frac{1}{N^{1/2}} \left\| \sum_{k=0}^{N-1} \Phi \circ f^k \right\|_{L^2(\mathbb{T}^2)} \tag{3}$$

Sketch of Effectiveness proof

Any sufficiently smooth function Φ with $\phi^{\perp} = 0$ is a smooth coboundary for a uniquely ergodic (irrational) skew-shift if and only if it is a measurable coboundary.

Assume that Φ is not a smooth couboundary, so that the lower bounds hold. Let $S_{\epsilon}^{\ell} \subset \mathbb{T}^2$ be the set defined as follows:

$$S_{\epsilon}^{\ell} := \{(x, y) \in \mathbb{T}^2 : |\Phi_{\ell}(x, y)| \geq \epsilon N_{\ell}^{1/2}\} . \tag{4}$$

From Upper and Lower bounds, one can show that there exist $\epsilon > 0$ and $\eta(\epsilon) > 0$ such that

$$\mathrm{Leb}(S_{\epsilon}^{\ell}) \geq \eta_{\epsilon} , \quad \text{for all } \ell \in \mathbb{N} . \tag{5}$$

If Φ were a measurable coboundary, this gives a contradiction. Thus Φ is not a measurable coboundary.

Along the sequence of the Theorem, the upper bound gives:

$$\|\Phi_{\ell}\|^2_{L^2(\mathbb{T}^2)} \leq M_s^2 \|\Phi\|_s^2 \mathrm{Leb}(S_{\epsilon}^{\ell}) N_l + \epsilon^2 N_{\ell}(1 - \mathrm{Leb}(S_{\epsilon}^{\ell})) .$$

From the Lemma: $c_{\Phi} N_{\ell} \leq M_s^2 \|\Phi\|_s^2 \mathrm{Leb}(S_{\epsilon}^{\ell}) N_{\ell} + \epsilon^2 (1 - \mathrm{Leb}(S_{\epsilon}^{\ell})) N_{\ell}$, hence $(c_{\Phi} - \epsilon^2) \leq (M_s^2 \|\Phi\|_s^2 - \epsilon^2) \mathrm{Leb}(S_{\epsilon}^{\ell})$.

Sketch of Effectiveness proof

Any sufficiently smooth function Φ with $\phi^\perp = 0$ is a smooth coboundary for a uniquely ergodic (irrational) skew-shift if and only if it is a measurable coboundary.

Assume that Φ is not a smooth couboundary, so that the lower bounds hold. Let $S_\epsilon^\ell \subset \mathbb{T}^2$ be the set defined as follows:

$$S_\epsilon^\ell := \{(x,y) \in \mathbb{T}^2 : |\Phi_\ell(x,y)| \geq \epsilon N_\ell^{1/2}\} . \tag{4}$$

From Upper and Lower bounds, one can show that there exist $\epsilon > 0$ and $\eta(\epsilon) > 0$ such that

$$\mathrm{Leb}(S_\epsilon^\ell) \geq \eta_\epsilon , \quad \text{for all } \ell \in \mathbb{N} . \tag{5}$$

If Φ were a measurable coboundary, this gives a contradiction. Thus Φ is not a measurable coboundary.

Along the sequence of the Theorem, the upper bound gives:

$$\|\Phi_\ell\|^2_{L^2(\mathbb{T}^2)} \leq M_s^2 \|\Phi\|_s^2 \mathrm{Leb}(S_\epsilon^\ell) N_l + \epsilon^2 N_\ell (1 - \mathrm{Leb}(S_\epsilon^\ell)) .$$

From the Lemma: $c_\Phi N_\ell \leq M_s^2 \|\Phi\|_s^2 \mathrm{Leb}(S_\epsilon^\ell) N_\ell + \epsilon^2 (1 - \mathrm{Leb}(S_\epsilon^\ell)) N_\ell$, hence $(c_\Phi - \epsilon^2) \leq (M_s^2 \|\Phi\|_s^2 - \epsilon^2) \mathrm{Leb}(S_\epsilon^\ell)$.

Sketch of Effectiveness proof

Any sufficiently smooth function Φ with $\phi^\perp = 0$ is a smooth coboundary for a uniquely ergodic (irrational) skew-shift if and only if it is a measurable coboundary.

Assume that Φ is not a smooth couboundary, so that the lower bounds hold. Let $S_\epsilon^\ell \subset \mathbb{T}^2$ be the set defined as follows:

$$S_\epsilon^\ell := \{(x,y) \in \mathbb{T}^2 : |\Phi_\ell(x,y)| \geq \epsilon N_\ell^{1/2}\}\,. \tag{4}$$

From Upper and Lower bounds, one can show that there exist $\epsilon > 0$ and $\eta(\epsilon) > 0$ such that

$$\mathrm{Leb}(S_\epsilon^\ell) \geq \eta_\epsilon\,, \quad \text{for all } \ell \in \mathbb{N}\,. \tag{5}$$

If Φ were a measurable coboundary, this gives a contradiction. Thus Φ is not a measurable coboundary.

Along the sequence of the Theorem, the upper bound gives:

$$\|\Phi_\ell\|^2_{L^2(\mathbb{T}^2)} \leq M_s^2 \|\Phi\|_s^2 \mathrm{Leb}(S_\epsilon^\ell) N_l + \epsilon^2 N_\ell (1 - \mathrm{Leb}(S_\epsilon^\ell))\,.$$

From the Lemma: $c_\Phi N_\ell \leq M_s^2 \|\Phi\|_s^2 \mathrm{Leb}(S_\epsilon^\ell) N_\ell + \epsilon^2 (1 - \mathrm{Leb}(S_\epsilon^\ell)) N_\ell\,,$ hence $(c_\Phi - \epsilon^2) \leq (M_s^2 \|\Phi\|_s^2 - \epsilon^2) \mathrm{Leb}(S_\epsilon^\ell)$.