Ergodicity of some open systems with particle-disk interactions

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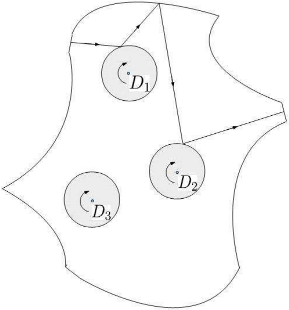
Corinaldo, Italy
June 3, 2010

Closed System



- Γ - domain with piecewise C^3 boundary.
- $D_1, \cdots, D_N$ pinned down disks:
  - angular positions $\varphi_1, \cdots, \varphi_N$
  - angular velocities $\omega_1, \cdots, \omega_N$
- Particle Collisions
  - with wall:
    $v'_\perp = -v_\perp$; $v'_t = v_t$;
  - with disks:
    $v'_\perp = -v_\perp$; $v'_t = R\omega$; $R\omega' = v_t$.[1]
  - particles do not interact with each other.

[1] particle-disk interactions introduced in [Klages, Nicolis, and Rateitschak 2000] and [Larralde, Leyvraz, and Mejía-Monasterio 2003]

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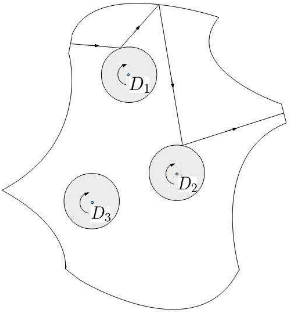

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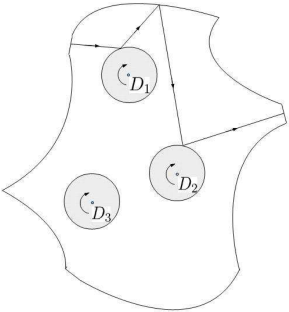

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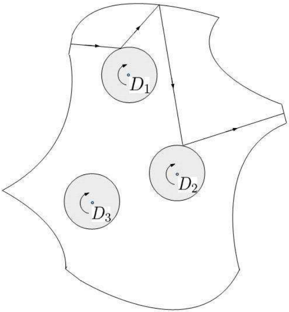

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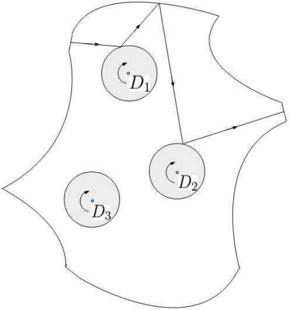

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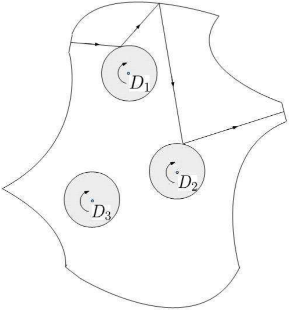

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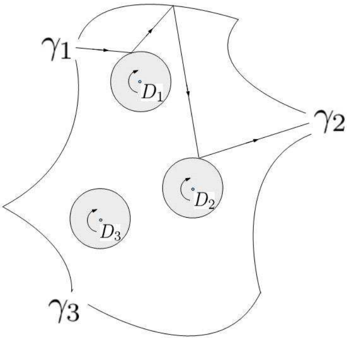

- $\gamma_1, \cdots, \gamma_l$ - openings.
- The system is coupled to a heat bath at each.
  - injection times exponentially distributed with rates $\varrho_1, \cdots, \varrho_l$;
  - distributions for positions $q_i \in \gamma_i$ and velocities $v_i \in H = \{(v_x, v_y : v_x > 0)\}$ are finite and positive everywhere.

Various geometric configurations considered in [Larralde, Leyvraz, and Mejía-Monasterio 2003], [Eckmann and Young 2006] [Eckmann and Jacquet 2007], and [Lin and Young 2010]

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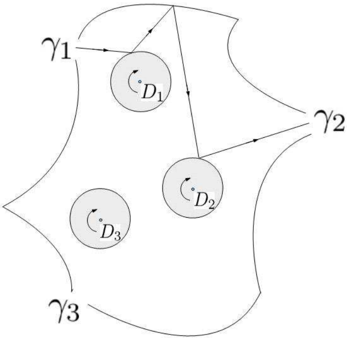

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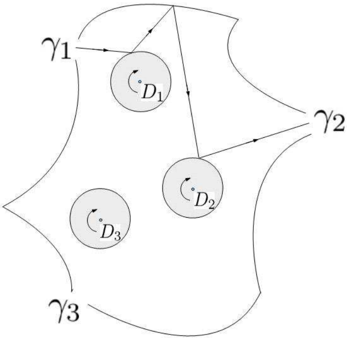

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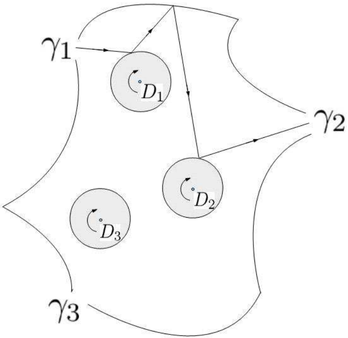

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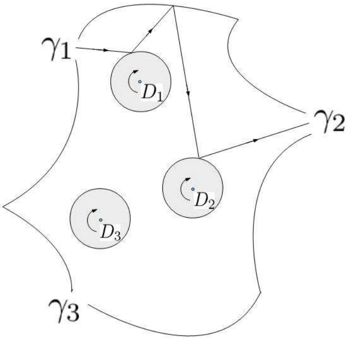

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The Phase Space

- All particles are identical and indistinguishable.
- The Phase Space

$$\Omega = \sqcup_k \Omega_k$$

- $\Omega_k = (\Gamma^k \times \partial D_1 \times \cdots \times \partial D_N \times \mathbb{R}^{2k+N})/\sim$
- invariant measure for the closed system with k particles:

$$m_k = (\lambda_2|_\Gamma)^k \times \rho|_{\partial D_1} \times \cdots \times \rho|_{\partial D_N} \times \lambda_{2k+N})/\sim .$$

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Invariant Measures

- Markov Process Φ_τ
  - no particles enter or exit: deterministic flow on Ω_k
  - a particle exits: jumps from Ω_k to Ω_{k-1}
  - a particle enters: jumps from Ω_k to Ω_{k+1}
- Invariant measures:
  - existence, uniqueness, ergodicity, absolute continuity with respect to m,
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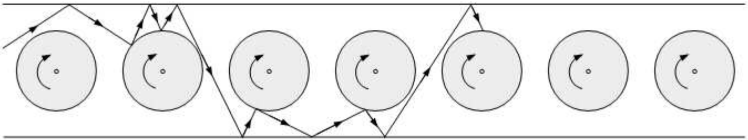
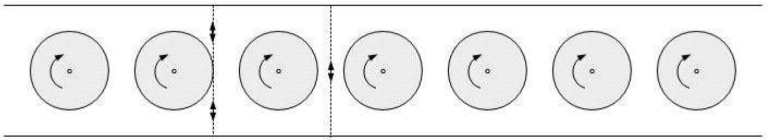
Results: Chain of Disks in a Rectangle



Geometry introduced in [Lin and Young 2010]

Special Case Results



- "trapped particles" (including stopped particles)

Theorem

If $\exists\mu$ - invariant measure with μ(states with "trapped particles") $= 0$

$$\Rightarrow$$

1. $\mu \ll m$ and
2. μ is ergodic.

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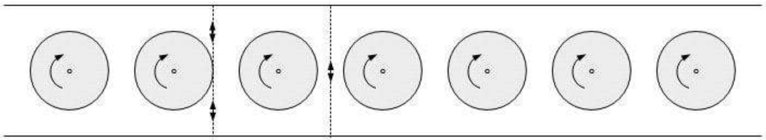

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Implications

Corollary 1

Any ergodic invariant measure ν can be written

$$\nu = \pi_k \times \mu,$$

- π_k *is a singular measure supported on k "trapped trajectories"*
- μ *is the absolutely continuous ergodic measure from the Theorem*

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If there exists any invariant measure, then there exists the unique absolutely continuous measure μ.

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Some ideas in the proof borrowed from [Balint, Lin, and Young 2010] and [Eckmann and Jacquet 2007]

- Admissible State = NO "trapped particles" +
  assuming no particles are injected,
  the first collisions of particles with disks (if any) are
  - NOT simultaneous with same disk and
  - NOT tangential.
- Let S be the set of all non-admissible states.

Lemma

μ-*inv.* + μ(*states with "trapped particles*) $= 0 \Rightarrow \mu(S) = 0$.

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Sample Paths and Canonical Neighborhoods

- Fix a sequence of particle injections $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ at times $0 < t_1 < t_2 < \cdots < t_n < T$,
  - $\epsilon_j = (t_j, \xi_j, v_j) \in [0, T] \times \cup\gamma_i \times H$.
- σ - a sample path on $[0, T]$ given ϵ and $X \in \Omega$ $=$ the path in the phase space Ω s.t.
  - σ starts from the state $X \in \Omega$
  - particles are injected according to ϵ.
- Σ - a *canonical neighborhood* of σ if
  - $\exists$ open neighborhoods U of X and $T_j \times Q_j \times V_j$ of (t_j, ξ_j, v_j) with nonintersecting $T_j \subset [0, T]$ s.t.
  - each sample path in Σ starts with an initial condition in U and exactly one particle is injected at some time in each T_j, with position in Q_j, and velocity in V_j.

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Intermediate Propositions

Proposition 1 ("ac measures stay ac")

$\nu \ll m \Longrightarrow (\Phi_t)_* \nu \ll m$ *for any* $t > 0$.

Proposition 2 ("acquiring density from Y_0")

$\exists$ $Y_0 \in \Omega_0$, U_0 *- nbhd of* Y_0*, time* T_0*, and* $A_0 \in \Omega_0$ *with* $m_0(A_0) > 0$*, s.t.*
- $\forall$ $Y \in U_0$, $[(\Phi_{T_0})_* \delta_Y]_{\ll}$ *has strictly positive density on* A_0*.*
- In particular, $[(\Phi_{T_0})_* \delta_Y]_{\ll}(\Omega) \neq 0$.

Proposition 3 ("getting from any admissible state to Y_0")

Given $X \in \Omega$ *admissible,* Y_0 *and* U_0 *from Proposition 2*
$\exists$ *time* T*, a sample path* $\sigma : X \to Y_0$ *on* $[0, T]$*, and a canonical neighborhood* Σ *of* σ*, s.t. each sample path in* Σ *ends in* U_0*.*

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Proof of the Theorem

- Propositions 2 and 3 $\implies \forall$ admissible state $X \in \Omega$, $\exists\, U$ - nbhd of X s.t. $\forall Y \in U$:
  - $[(\Phi_{T+T_0})_* \delta_Y]_{\ll}$ has strictly positive density on A_0
  - in particular, $[(\Phi_{T+T_0})_* \delta_Y]_{\ll}(\Omega) \neq 0$.

$(\Phi_t)_*\mu = [(\Phi_t)_*(\mu_{\ll})]_{\ll} + [(\Phi_t)_*(\mu_{\ll})]_{\perp} + [(\Phi_t)_*(\mu_{\perp})]_{\ll} + [(\Phi_t)_*(\mu_{\perp})]_{\perp}$

- Assume $\mu_{\perp}(\Omega) \neq 0$.
  - $\mu_{\perp}(S) = 0$ + fact above + Prop 1 ("ac measures stay ac") $\implies \forall t > T + T_0$, $[(\Phi_t)_*\mu_{\perp}]_{\ll}(\Omega) \neq 0$.
  - Prop 1 ("ac measures stay ac") $\implies \forall t > 0$, $[(\Phi_t)_*(\mu_{\ll})]_{\perp}(\Omega) = 0$.
  - $\implies \forall t > T + T_0$, $[(\Phi_t)_*\mu]_{\ll}(\Omega) > \mu_{\ll}(\Omega)$, which contradicts the invariance of μ.

$$\implies \mu \ll m.$$

- For all admissible states, the ergodic averages are equal for measure 1 sets of sample paths.

$$\implies \mu \text{ is ergodic.}$$

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Proof of Proposition 1

Proposition 1 ("ac measures stay ac")

$\nu \ll m \Longrightarrow (\Phi_t)_* \nu \ll m$ *for any* $t > 0$.

- No particles exit or enter: measure stays a.c. since m_k is.
- When a particle exits, the measure projects to Ω_{k-1} and stays a.c.
  - Ω_{k+1} is 4D larger than Ω_k.
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    1D - time, 1D - position, 2D - velocity.
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Proposition 2

Given $Y_0 \in \Omega_0$ $\omega_j \neq 0$ $\forall j$,
$\exists$ *nbhd* U_0 *of* Y_0, *time* T_0, *and* $A_0 \in \Omega_0$ *with* $m_0(A_0) > 0$, *s.t.*
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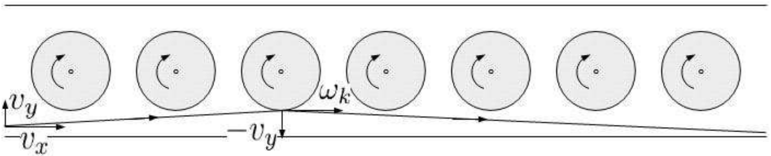

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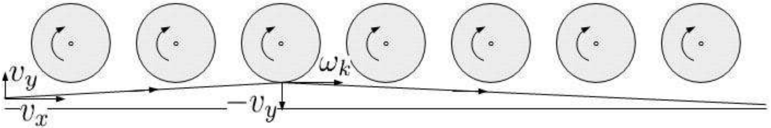

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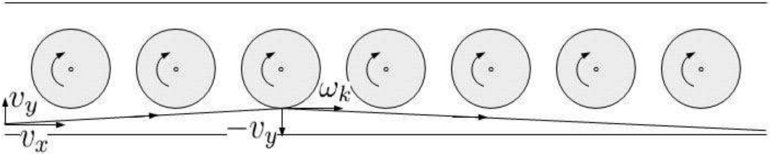

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Proof of Proposition 3

Proposition 3 ("getting from any admissible state to Y_0")

Given $X \in \Omega$ admissible, Y_0 and U_0 from Proposition 2, $\exists$ time T, a sample path $\sigma : X \to Y_0$ on $[0, T]$, and a canonical neighborhood Σ of σ, s.t. each sample path in Σ ends in U_0.

1. Flush particles out: $\sigma_X : X \to X_0 \in \Omega_0$.
2. Go from any X_0 to Y_0: $\sigma_0 : X_0 \to Y_0$
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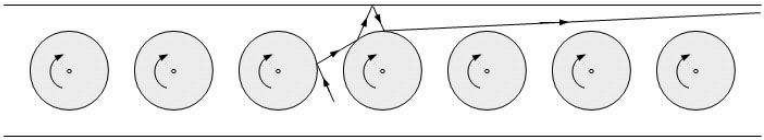
Flush Particles Out: 1 particle

- Proper projected particle path:
  - finite number of straight segments that meet on $\partial\Gamma$
  - meet at wall: incoming and outgoing angles are equal
  - meet at disk: any angles except $\pm\frac{\pi}{2}$



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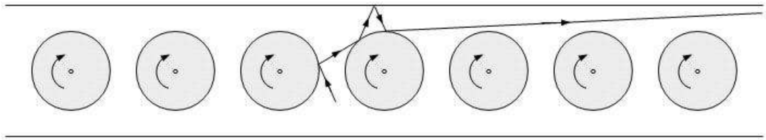

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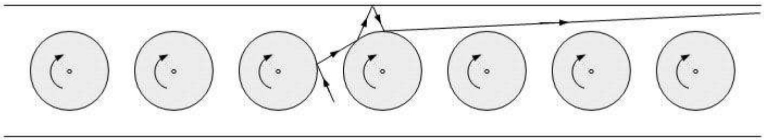

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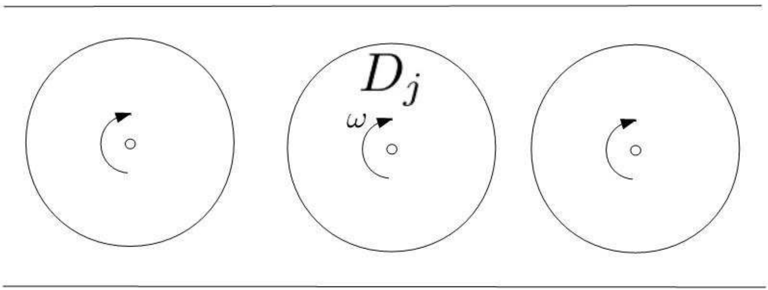

Controllability Lemma

ω - angular velocity of D_j; no particles.
Given ω' and time τ, there exists σ on $[0, \tau]$ s.t.
- *at time τ, D_j has ang. vel. ω' and no particles.*
- *on $[0, \tau]$, all particles follow admissible paths and only hit disks $D_1, \cdots, D_{j-1}$ except one collision with D_j.*

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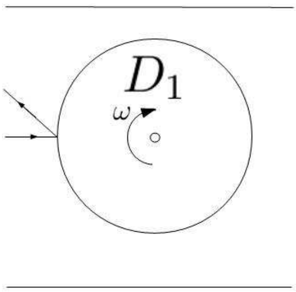

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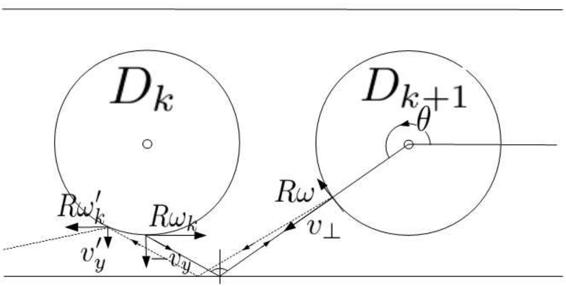

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- The Main Lemma $\Longrightarrow$ can follow any proper projected particle path.
- Many particle system:
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2: From X_0 to Y_0

Lemma

$X_0, Y_0 \in \Omega_0$:

- $X_0 : (\varphi_1, \omega_1), \cdots, (\varphi_N, \omega_N)$
- $Y_0 : (\varphi'_1, \omega'_1), \cdots, (\varphi'_N, \omega'_N)$

Given time T, there exists a sample path $\sigma_0 : X_0 \to Y_0$ such that all particles follow admissible paths.

- Proof: application of the Main Lemma $2N$ times.

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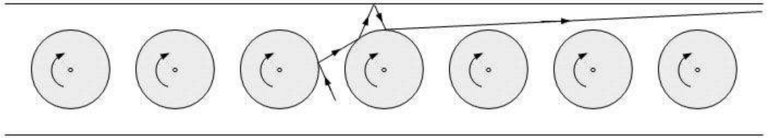
- $Y_0 : (\varphi'_1, \omega'_1), \cdots, (\varphi'_N, \omega'_N)$

Given time T, there exists a sample path $\sigma_0 : X_0 \rightarrow Y_0$ such that all particles follow admissible paths.

- Proof: application of the Main Lemma $2N$ times.

3:Canonical Neighborhood of σ



- Near a proper projected particle path, particle's final position and velocity depend continuously on its initial position and velocity and angular velocities of the disks it collides with.
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- If in $\sigma : X \to Y_0$ all particles follow admissible paths, $\exists$ a canonical neighborhood Σ of σ s.t. each sample path in Σ ends in U_0.

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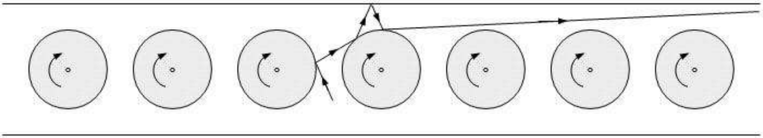

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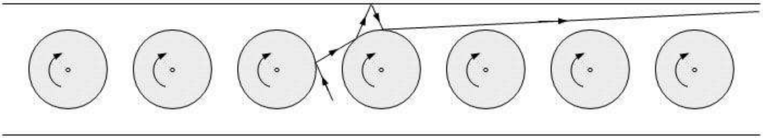

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Conclusion

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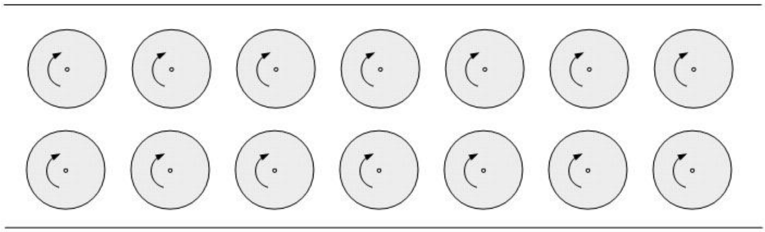
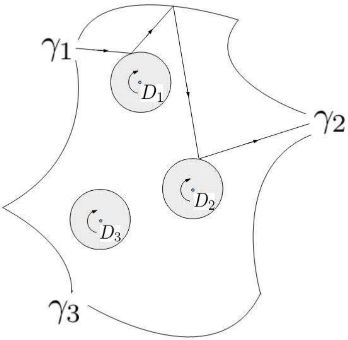
Can generalize to systems with similar geometries, e.g.



Other Geometries



- The Theorem applies to systems with other geometries with only slight modifications.
- Things to check:
  - Can flush particles out: proper projected particle path from any point on any disk to an exit.
  - Controllability Lemma applies: proper projected particle path from an opening to a disk that meets that disk radially

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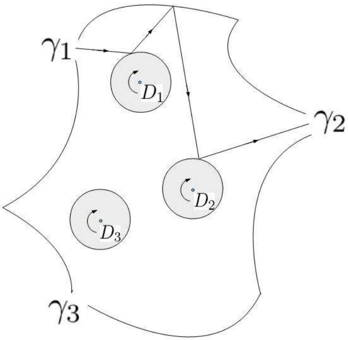

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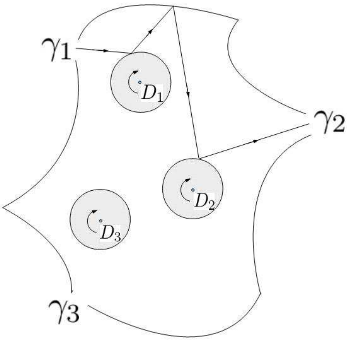

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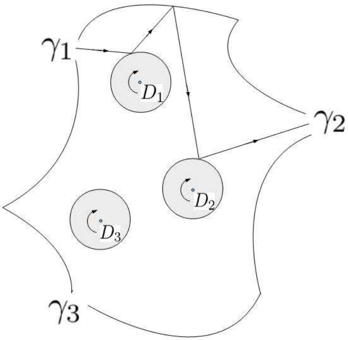

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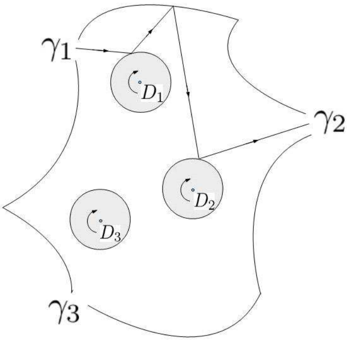

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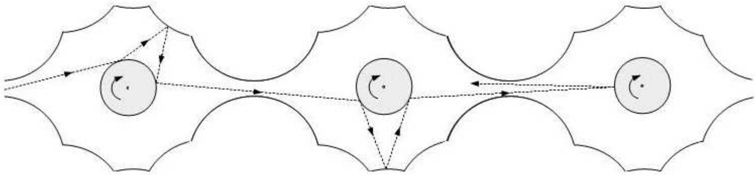
Chains and Lattices of Illuminated Cells



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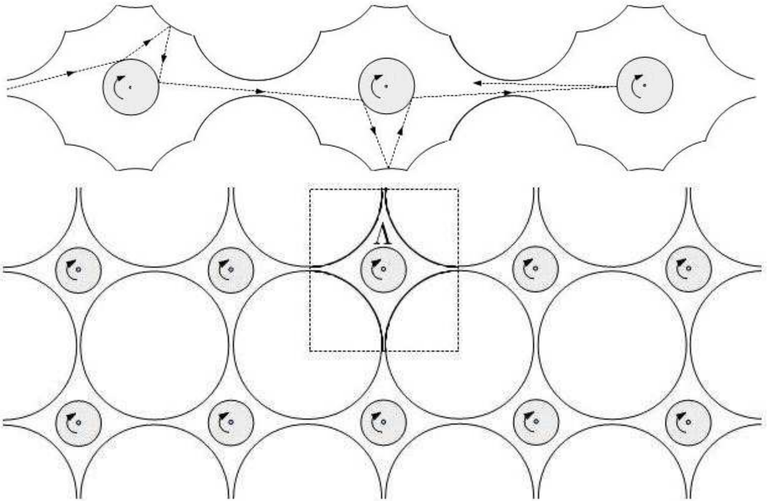
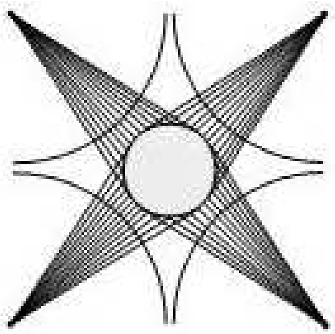
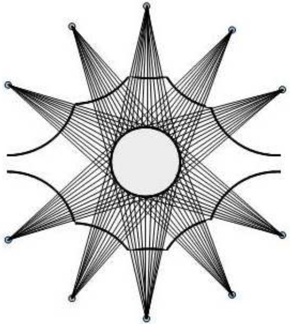


Illuminated Cells



(a) an illuminated cell



(b) an illuminated cell

Figure: Illumination Property

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