Corinaldo, Italy

Dynamics of Periodically-Kicked Oscillators

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Outline of talk

I. A 2D linear shear flow
II. Periodically kicked oscillators (general) } *
III. " " Hopf bifurcations (eg. reaction-diffusion eqtns)
IV. White-noise forcing (numerics)

Focus : phenomena

Co-authors : *Qiudong Wang, Kening Lu, Kevin Lin

* review article (w/ K Lin), vol in honor of Smale's 80th
J Fixed Pt Theory & Appl, 2010

I. Periodic kicking of 2D linear shear flow

$$\begin{cases} \dot{\theta} = 1 + \sigma y \\ \dot{y} = -\lambda y \end{cases} \quad + A \sin(2\pi\theta) \cdot \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

[Z] '78

[WY] 2002

[LY] 2008

$(\theta, y) \in S^1 \times \mathbb{R}$, σ, λ, A, τ = constants

- Unforced eqtn : Φ_t = flow, $\gamma = \{y = 0\}$ limit cycle
- Forced eqtn : $\Psi_\tau = \Phi_\tau \circ \kappa$,

$$\kappa(\theta, y) = (\theta, y + A\sin(2\pi\theta)) \quad \text{"kick map"}$$

Parameters of interest : $\begin{cases} \sigma = \text{shear} \\ \lambda = \text{damping} \\ A = \text{amplitude of kicks} \\ \tau = \text{time interval betwn kicks} \end{cases}$

Geometry of Ψ_τ : dependence on σ, λ, A ($\lambda\tau$)

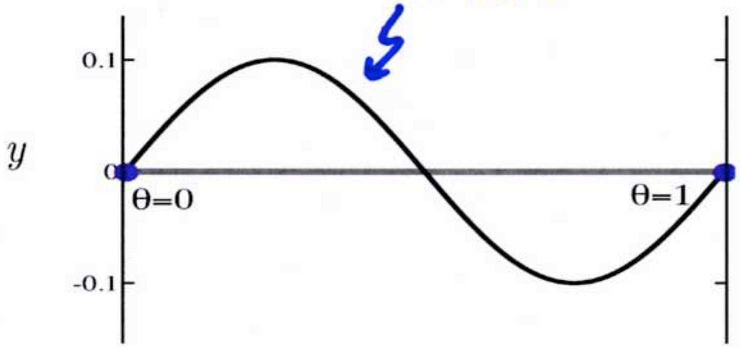
$\lambda = A = 0.1$

$\tau = 10$

$\sigma \uparrow$

$$\begin{cases} \dot{\theta} = 1 + \sigma y \\ \dot{y} = -\lambda y + A \sin 2\pi\theta \sum \delta(t - n\tau) \end{cases}$$

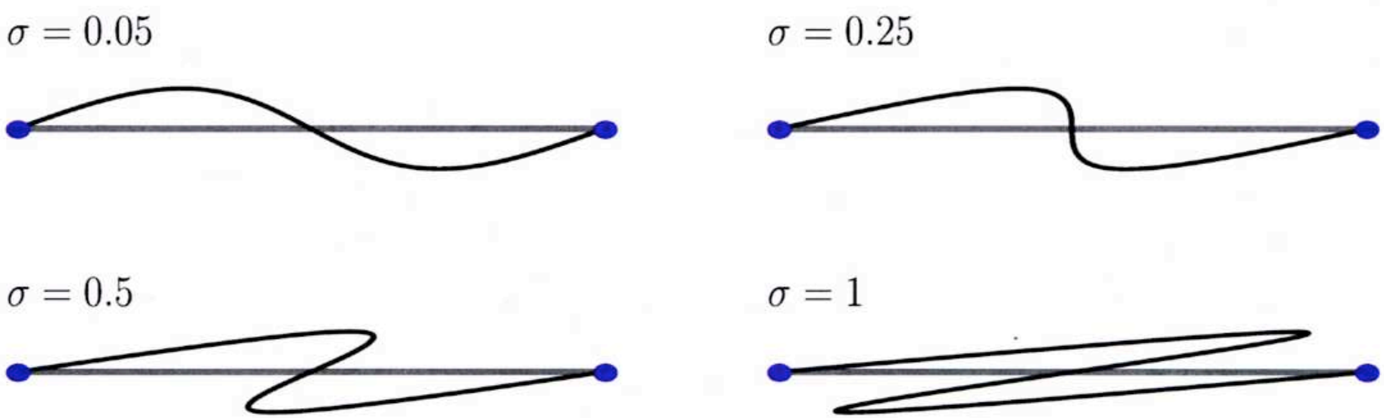
$\kappa(\gamma)$



(a) The limit cycle γ and its image $\kappa(\gamma)$ after one kick

Fix σ, $\lambda \uparrow$

Similar pictures backwards w/ $\lambda\tau$ fixed



(b) $\Psi_\tau(\gamma)$ for $\tau = 10$

$\Psi_\tau(\gamma) = \Phi_\tau \circ \kappa(\gamma)$ for $\sigma = 0.005, 0.25, 0.5, 1$

Basic facts

- $$\Psi_\tau : \begin{pmatrix}\theta \\ y\end{pmatrix} \mapsto \begin{pmatrix}\theta + \tau + \frac{\sigma}{\lambda}\left[y + A\sin(2\pi\theta)\right]\cdot\left(1-e^{-\lambda\tau}\right) \\ e^{-\lambda\tau}\left[y + A\sin(2\pi\theta)\right]\end{pmatrix} \pmod 1$$

  $\mathrm{Det}(D\Psi_\tau) \equiv e^{-\lambda\tau}$ $\leftarrow$ *assume* $<$ e.g. $\frac{1}{2}, \frac{2}{3}$

- Key is $\boxed{\frac{\sigma}{\lambda} A = \frac{\text{shear}}{\text{contraction rate}} \cdot \text{kick amplitude}}$ when $e^{-\lambda\tau} \ll 1$ (so $y \approx 0$)

  Effect of $\frac{\sigma}{\lambda} A$ attenuated by $\left(1-e^{-\lambda\tau}\right)$

- Trapping regions e.g. $U = \{ |y| \le A(e^{\lambda\tau}-1)^{-1} \}$

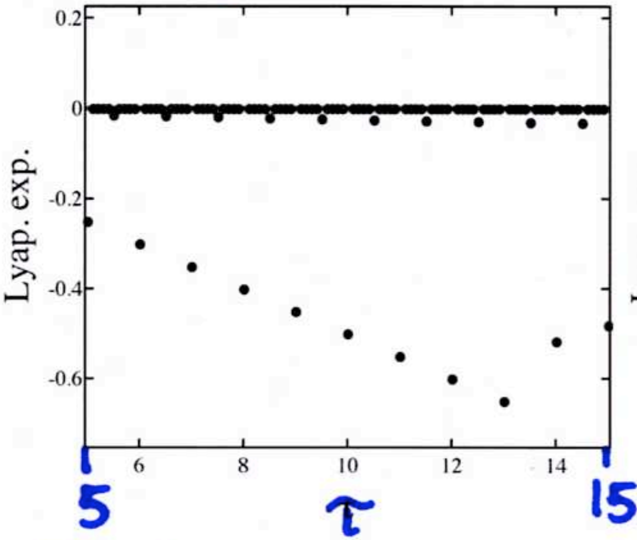
  $$\Gamma \stackrel{\text{def}}{=} \bigcap_{n\ge 0} \Psi_\tau^n(U) \quad \text{attractor}$$

Lyapunov exponents of Ψ_τ

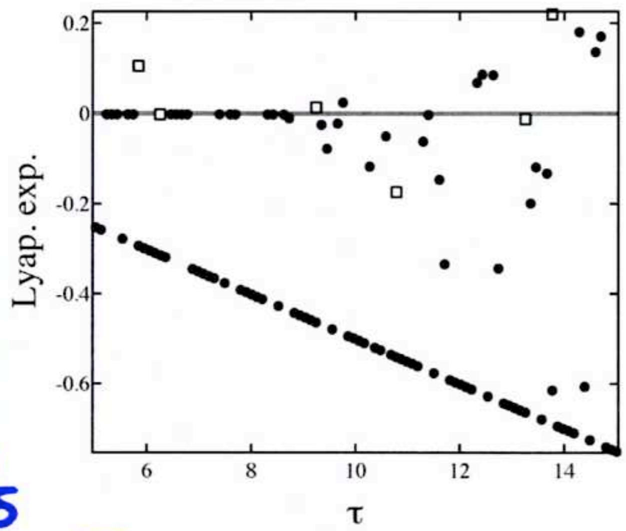
$\lambda = A = 0.1$

$$\Lambda_{max}(z) = \lim_{n\to\infty} \frac{1}{n} \log \| D\Psi_\tau^n(z) \|$$

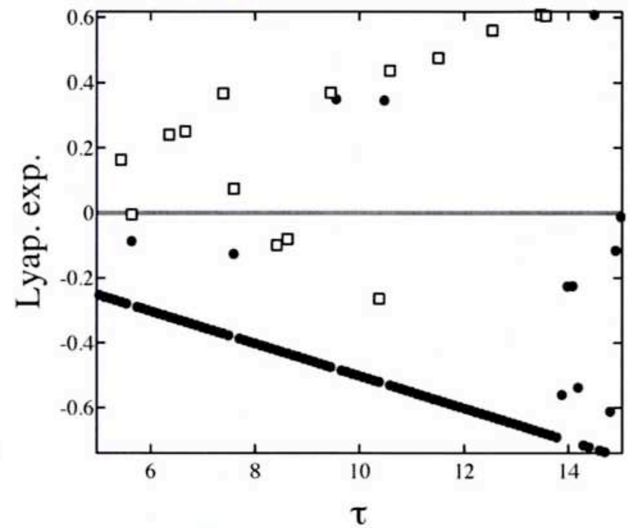
$\sigma = 0.05$



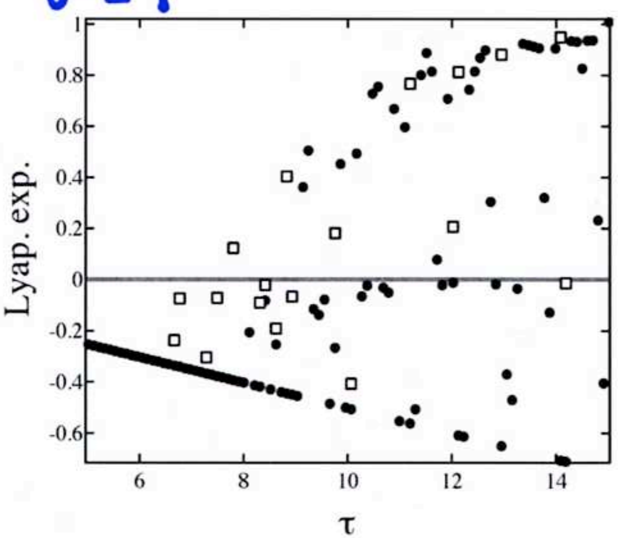
$\sigma = 0.25$



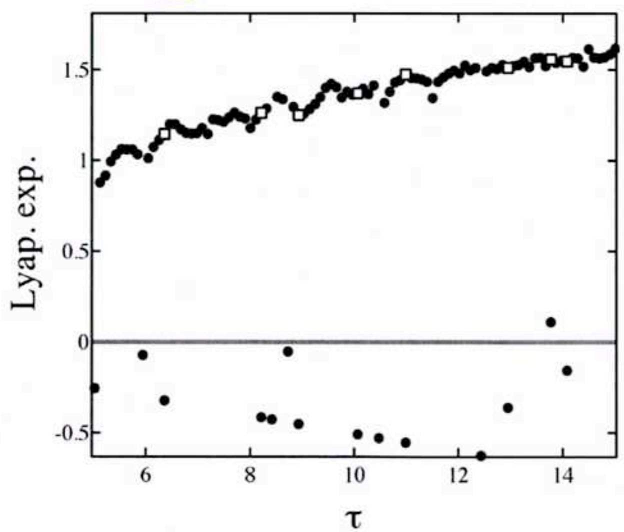
$\sigma = 0.5$



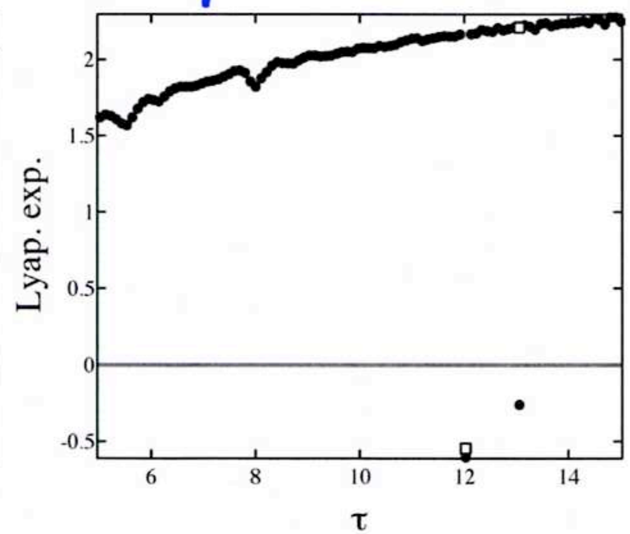
$\sigma = 1$



$\sigma = 2$



$\sigma = 4$



Next: Identify prominent geometric structures

Low-shear regimes

Prop 1 $\frac{\sigma}{\lambda}$, A suff small $\Rightarrow$

- Γ = closed invariant curve near $\{y=0\}$, attracts all pts
- Dynamics of Ψ_τ on Γ:

rotation # $\in \mathbb{Q}$	rot. # $\notin \mathbb{Q}$
sinks/sources on Γ	conj. to rigid rotation
open, dense(?) parameters	pos. Leb meas set

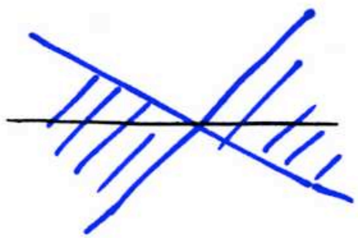
Prop 2 Increasing $\frac{\sigma}{\lambda}$, A $\Rightarrow$
invariant circle breaks, sink(s) appears

Breaking of invariant curves

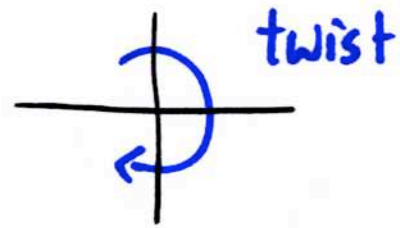
$$D\Psi_\tau(\theta,y) = \begin{pmatrix} 1 + 2\pi \frac{\sigma}{\lambda} A\cos(2\pi\theta)(1-e^{-\lambda\tau}) & \frac{\sigma}{\lambda}(1-e^{-\lambda\tau}) \\ e^{-\lambda\tau}\, 2\pi A\cos(2\pi\theta) & e^{-\lambda\tau} \end{pmatrix}$$

$$\det(D\Psi_\tau) \equiv e^{-\lambda\tau}$$

Low shear: inv cones



$\frac{\sigma}{\lambda}, A \uparrow$,



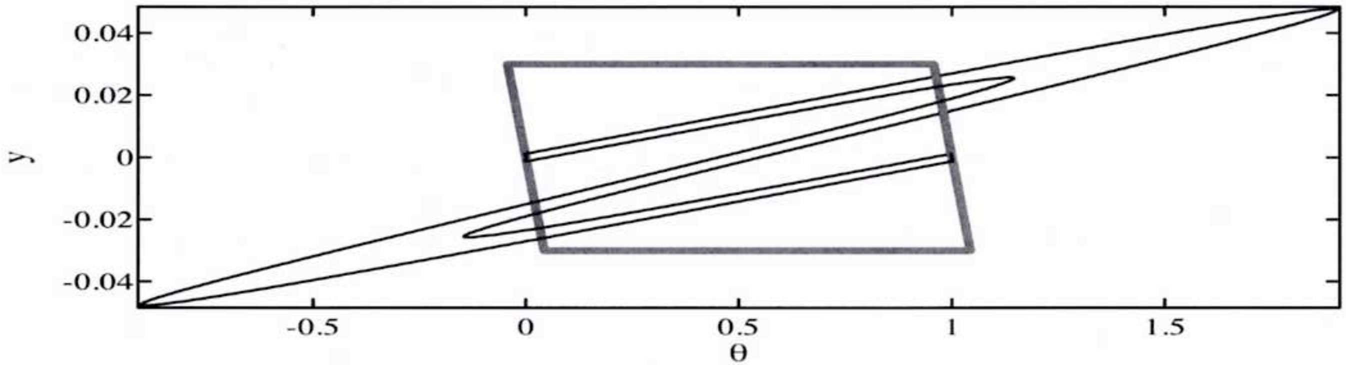
strongest at $\theta = \frac{1}{2}$
(where $\cos 2\pi\theta = -1$)

$\Rightarrow$ expect breaking of $\odot$ to occur first for $\tau \in \mathbb{Z}$
at $(\theta, y) = (\frac{1}{2}, 0)$ = fixed pt of Ψ_τ
as e.v. become complex conj, $|\cdot| = e^{-\frac{1}{2}\lambda\tau}$

Rmks 1. sink robust (no bif possible for cx e.v.)
2. influence local, i.e. other events possible

Higher-shear regimes

[**Prop** $\frac{\sigma}{\lambda} A >$ const $\Rightarrow$ horseshoes present.



Remarks

1. Horseshoes $\Rightarrow$ existence of chaotic orbits, not nec. "typical"
2. Sinks near homoclinic tangencies (Newhouse)
3. Expansion + $|\det D\Psi_\tau| < 1 \Rightarrow$ contraction also present
   $\Lambda_{max} > 0$ or < 0 dep on $\|D\Psi_\tau^n(v)\| \sim ?$ *on balance*

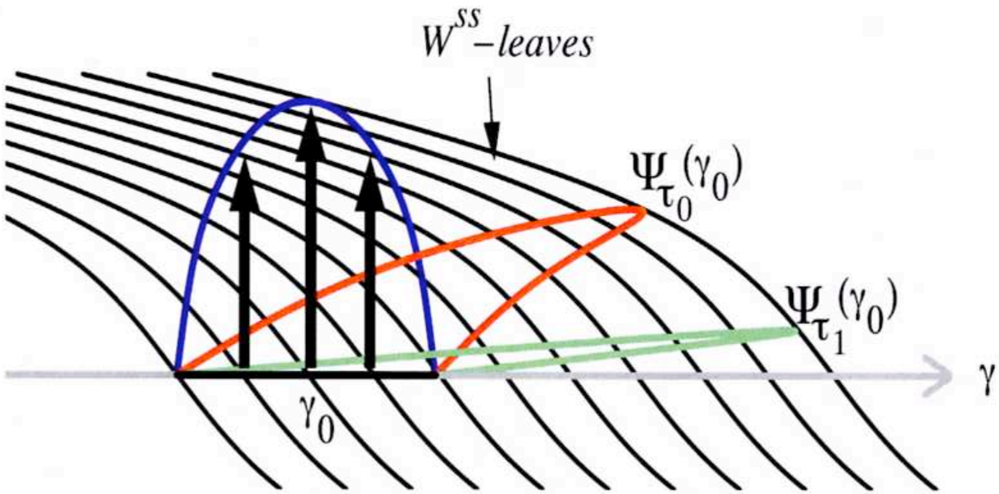
   $\Rightarrow$ competition betwn $\underbrace{\text{horseshoes + sinks}}_{\Lambda_{max} < 0 \,:\, \text{transient chaos}}$ vs $\underbrace{\text{"strange attractors"}}_{\Lambda_{max} > 0}$

   wins as $\frac{\sigma}{\lambda} A \uparrow$

The general kicked-oscillator picture

Setting $\begin{cases} \text{ODE}, \ \Phi_t, \ n\text{-dim manifold}, \ \gamma = \text{hyperbolic periodic orbit} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{period} = p \\ \Psi_\tau = \Phi_\tau \circ \kappa, \ \ \kappa = \text{kick map} \end{cases}$

Assume $\exists\, U \subset$ basin of γ s.t. $\kappa(U) \subset U$



Analysis

W^{ss} = strong stable foliation of Φ_t in U

i.e. $W^{ss}(x) = \{ y : d(\Phi_t(y), \Phi_t(x)) \to 0 \}$

Then if $\tau = np + a$, $n \in \mathbb{Z}^+$,

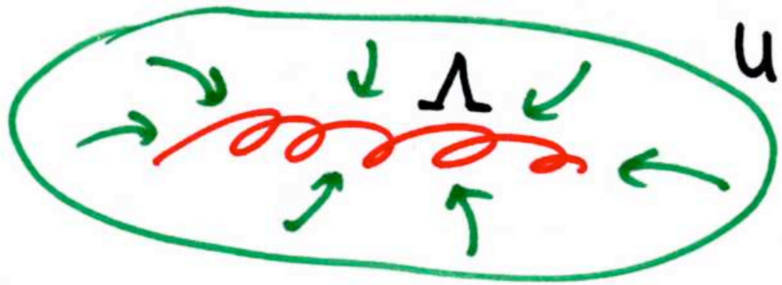
$\Psi_\tau = \Phi_a \circ$ (slide along W^{ss})

Folding ($\Rightarrow$ chaos) determined by
(i) kick direction in rel to W^{ss}, (ii) kick strength

e.g. kicks along W^{ss}, or permuting W^{ss} : no effect

Prev. Ex : slope$(W^{ss}) = -\frac{\lambda}{\sigma}$

Ergodic theory of "Strange attractors"



Λ = attractor

U = basin of attraction

(Assume vol decreasing)

- Uniformly hyperbolic (Axiom A) attractors : (Sinai, Ruelle, Bowen 70's)

  $\exists$ an invariant prob meas μ (called SRB measure) on Λ

  s.t. $\frac{1}{n}\sum_{0}^{n-1}\varphi(f^i x) \to \int \varphi \, d\mu$, Leb-a.e. $x \in U$.

  In particular, $\lambda_{max} > 0$ Leb-a.e. $x \in U$

- Extension of SRB meas & related to general setting (Ledrappier-Young, Pugh-Shub)

  but existence not guaranteed (balancing.....)

Rmks 1. SRB is only known way to guarantee $\Lambda_{max} > 0$

2. Outside of Axiom A (no inv cones), existence of SRB known only for "rank-one attractors"

Theory of rank-one attractors (Wang-Young)

Informal defn : $f : M \circlearrowleft$ has rank one if

Df = rank 1 as linear map $+ \mathcal{O}(\varepsilon)$ [no inv cones]

Theorem (2D: CMP 2001, n-D: Annals 2008)

$F_{a,\varepsilon}$ = 1-parameter family of maps for each $\varepsilon > 0$

Assume 1. $F_{a,\varepsilon} \to F_{a,0}$ as $\varepsilon \to 0$, ie. singular limit well defined

2. $F_{a,0}$ sufficiently expanding + technical conds (transversality, nondegeneracy)

Then $\forall\, \varepsilon > 0$ suff small, $\exists\, \Delta(\varepsilon)$ = pos meas set of a

s.t. (i) $F_{a,\varepsilon}$ has ergodic SRB meas

(ii) $\Lambda_{max} > 0$ Leb-a.e. in basin

Rmks. $\Delta(\varepsilon)$ not open; sinks for some $F_{a,\varepsilon}$, $a \notin \Delta(\varepsilon)$

- Borrowed techniques from Benedicks-Carleson on Hénon maps

Application of rank-one theory to kicked oscillators

$\Psi_\tau = \Phi_\tau \circ \kappa$, $\quad \tau = np + a$, $\; a \in [0,p)$, $\; p =$ period of γ

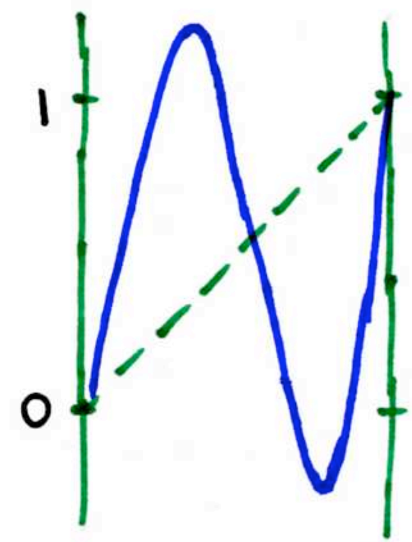
$F_{a,\varepsilon} \stackrel{\text{def}}{=} \Psi_{np+a} = \Phi_a \circ (\Phi_{np} \circ \kappa)$, $\quad \varepsilon = e^{-\lambda np}$, $\; \lambda =$ min contraction to γ

As $\varepsilon \to 0$, $F_{a,\varepsilon} \to F_{a,0} = \Phi_a \circ \pi^{ss}$

where π^{ss} = sliding along W^{ss} to γ

2D linear model

$$F_{a,0} : \begin{pmatrix} \theta \\ y \end{pmatrix} \mapsto \begin{pmatrix} \theta + a + \frac{\sigma}{\lambda} A \sin(2\pi\theta) \\ 0 \end{pmatrix}$$



"Sufficiently expanding" for $\frac{\sigma}{\lambda} A$ medium large

Remarks on analytic techniques

- Rank-one theory requires $\varepsilon = e^{-\lambda\tau} \ll 1$
- Numerically : OK for e.g. $\varepsilon > \frac{2}{3}$ [Lin-Young]

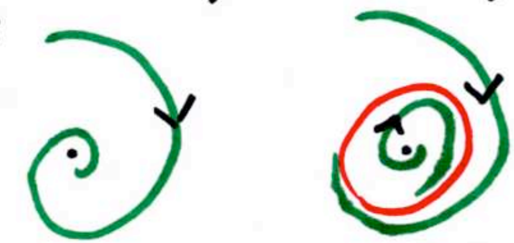
conceptual breakthrough needed / computational tool a "must" for dyn syst

Shear-induced chaos occurs in many forms, e.g.

Periodic forcing of systems undergoing generic supercrit. Hopf bifurcations

- Unforced eqtn: $\dot{x} = F_\mu(x)$, $x \in \mathbb{R}^n$, μ = parameter, $F_\mu(0) = 0 \ \forall \mu$

  At $\mu = 0$, leading cx conj e.v. lose stability:



  Normal form:

  $$\dot{z} = k_0(\mu) z + k_1(\mu) z^2 \bar{z} + \text{h.o.t.}$$

  "twist" $\tilde{T} \stackrel{\text{def}}{=} \dfrac{\operatorname{Im} k_1(0)}{-\operatorname{Re} k_1(0)}$

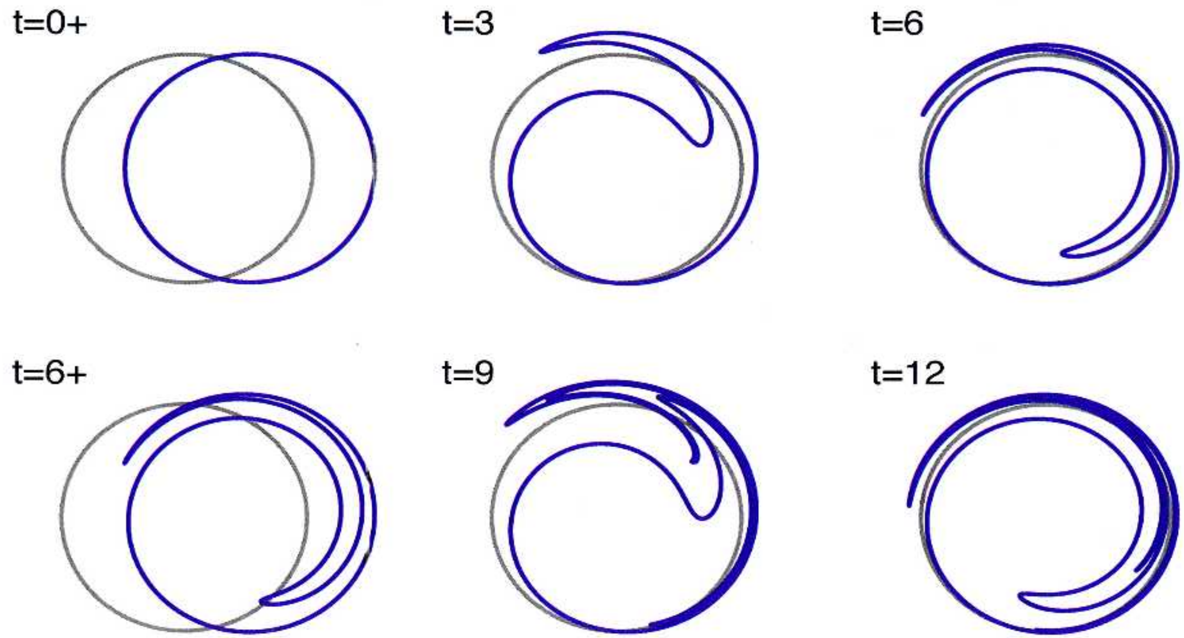
- Kick $= \kappa$. Assume $|\pi^c \kappa(0)| = \rho \neq 0$, π^c = proj $\langle$leading eigen dir$\rangle$

Theorem (Lu-Wang-Young) For $0 < \mu \ll 1$,
if $|\tilde{T}| \cdot \dfrac{\rho}{\sqrt{\mu}}$ suff large & $\tau \gg 1$,
then Ψ_τ has "strange attractor" for pos meas set of τ.

$\Psi_\tau = \Phi_\tau \circ \kappa$

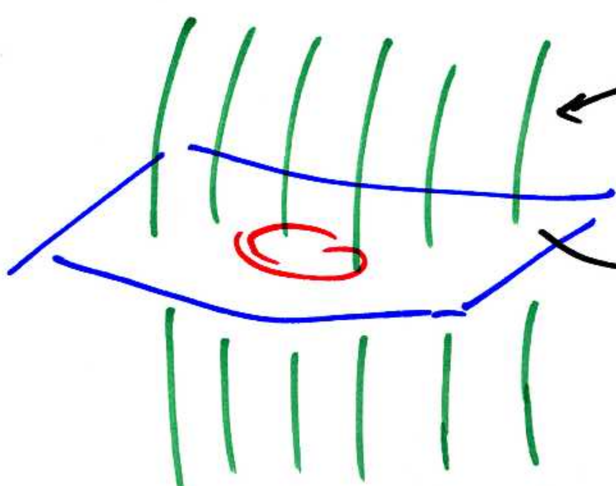
$\tilde{T}\left(\frac{\rho}{\sqrt{\mu}}\right) \longleftrightarrow \frac{\sigma}{\lambda}(A)$ ρ = kick size , $\sqrt{\mu}$ = radius of limit cycle

$\tilde{T} = 10,\ \tau = 6$:



[LWY] -

Hopf theorem proved for system on Hilbert space



W^{ss} = codim 2 strong stable foliation for Φ_τ

W^c = 2D center mfld

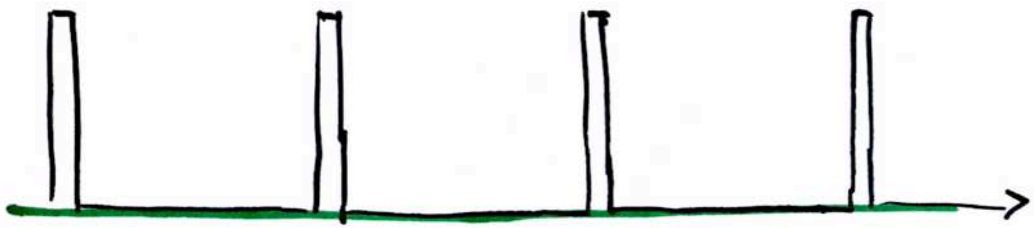
Application to dissipative evolutionary PDEs

such as

$\Omega \overset{\text{bdd}}{\subset} \mathbb{R}^d$, $u \in \mathbb{R}^m$, D = positive diagonal matrix

$$\begin{cases} u_t = D\Delta u + f_\mu(u) \ \vdots + \rho\varphi(x)P_\tau(t) & x \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega \end{cases}$$

$P_\tau(t)$ = "pulse"
period τ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Ex The Brusselator

$$\begin{cases} u_t = d_1 \Delta u + a - (b+1)u + u^2 v \\ v_t = d_2 \Delta v + bu - u^2 v \end{cases}$$

Dirichlet or Neumann

u, v = chemical concentrations ; a, b = initial substances

Known to have Hopf, arb large "twist"

$\Longrightarrow$ strange attractors etc. w/ e.g. $\varphi(x) = \sin \pi x$

Stochastic forcing

$$dx_t = a(x_t)\,dt + \sum_{i=1}^{k} b_i(x_t) \circ dW_t^i$$

$x \in$ manifold, W_t^i = indep standard Brownian motion

Facts (stochastic flows) :

- $\{F_{\omega;t_1,t_2}\}_{t_1<t_2}$ well defined a.e. ω
- Lyapunov exponents nonrandom

Linear shear model : driven by white noise

$$\begin{cases} \dot{\theta} = 1 + \sigma y \\ \dot{y} = -\lambda y \quad + \boxed{a \sin(2\pi\theta)\, dW_t} \end{cases}$$

Parameters : σ, λ, a in place of A, τ

Numerical finding (Lin-Young) : **SAME** , except

- Λ_{max} varies continuously (smoothly?) w/ parameter
- a can be v. small, eg. $\sigma = 3$, $\lambda = 0.5$, $a = 0.05$, for $\Lambda_{max} > 0$